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Volume 8 JANUARY 15, 1922 Number 1

THE RELATION OF CHALCEDONY TO THE OTHER FORMS OF SILICA

BY EDWARD W. WASHBURN AND LOUIS NAVIAS

DEPARTMENT OF CERAMIC ENGINEERING, UNIVERSITY OF ILLINOIS

Communicated by W. A. Noyes, Nov. 21, 1921

Silica is known to occur in three distinct enantiotropic forms, each form having two or more enantiotropic sub-forms. These forms are quartz (α and β), tridymite (α, β_1, and β_2) and cristobalite (α and β) to which may be added silica glass, the amorphous or liquid form. The various transition and fusion temperatures and the ranges of stable existence of all of these forms have been accurately determined by Fenner.[1] The position of the mineral chalcedony in this system has, however, never been satisfactorily established, although Fenner seemed inclined to view it as a form different from any of the preceding.

The present investigation is a contribution to the elucidation of the chalcedony puzzle. The method of investigation adopted was a careful determination of a number of physical properties of the raw and calcined chalcedony and a comparison of these properties with those of the four recognized forms of silica. Two minerals belonging to the chalcedony type were used. One of these was a very pure typical chalcedony from Yellowstone Park and the other was a black pebble of French flint.

For comparison a summary of all the data obtained with these materials together with the corresponding data for the four recognized forms of silica is displayed in the accompanying table.

The density data were determined on the finely ground powder by the pycnometer method, the pycnometer and powder being heated to 400° C and evacuated with a mercury vapor pump for several hours before admitting the air-free distilled water. After weighing, a sample of the powder was removed from the pycnometer and while still under water was sent to the Geophysical Laboratory and subjected to a pressure of 1000 atmospheres for two hours. This treatment resulted in a density increase of only six units in the fourth decimal place.

Summary of Results

Results in black face type were obtained in this investigation. After calcination at 1450° for 2 hours the chalcedony analysed 99.87 and the flint 99.6% SiO_2. Inversion temperatures were obtained from cooling curves and volume changes with a dilatometer.

	RAW CHALCEDONY	RAW FLINT	QUARTZ		CALCINED CHALCEDONY		CALCINED FLINT		CRISTOBALITE		TRIDYMITE		SILICA GLASS
			α		α	β	α	β	α	β	α	β	
Specific gravity, 25°/25°	2.55–2.61	2.61–2.63	2.65		**2.175**	. . .	**2.25**	. . .	**2.33**	. . .	**2.27**	. . . .	2.194–2.213
Index of refraction	**1.533–1.538**	**1.533–1.539**	**1.544–1.553**		**1.456–1.470**	. . .	**1.483**	. . .	**1.484**	. . .	**1.475**	. . . .	**1.457–1.460**
Inversion temperature t_I			575°	570°	. . . .	**220°**	. . .	**227°**	. . .	**244°**	**117°**	. . . .	None
Coefficient of cubical expansion near t_I													
(a) Per cent per degree × 10^2			0.65	−0.12	**2.7 ± 0.6**	**2.8 ± 0.4**	**1.9 ± 0.8**	**3.0 ± 0.6**	**2.4 ± 1.1**	**3.2 ± 0.8**	**0.3 ± 0.2**	**0.8 ± 0.2**	
(b) Cc. per gram per deg. × 10^4			0.25	−0.049	**1.3 ± 0.3**	**1.4 ± 0.2**	**0.9 ± 0.4**	**1.4 ± 0.3**	**1.1 ± 0.5**	**1.5 ± 0.4**	**0.15 ± 0.12**	**0.38 ± 0.08**	
Volume increase on inversion $\alpha \longrightarrow \beta$ at t_I													
(a) Per cent			2.16		**2.63**		**3.09**		**2.83**		**0.142**		
(b) Cc per gram × 10^4			84		**131**		**150**		**132**		**6.3 ± 0.6**		

The inversion temperatures, coefficients of expansion and volume changes on inversion were obtained with a dilatometer holding 60 grams of powder and filled *in vacuo* with a mixture of H_2SO_4 and K_2SO_4 as the expansion liquid. The indices of refraction were measured with a petrographic microscope, using the Becke line method. The values given represent the upper and lower limits, that is, no part of the material showed an index outside of the limits given.

A study of the data given in the table brings out the following facts: (1) The density of raw flint and chalcedony is appreciably lower than, but close to, that of quartz, the difference being not greater than that which might arise from moisture and other impurities in the raw material. The density of the calcined flint is slightly lower but close to that of tridymite. The density of the calcined chalcedony is distinctly lower than that of silica glass, the lightest known form of silica.

(2) The indices of refraction of the raw flint and chalcedony are substantially the same and while close to that of quartz are distinctly lower. The index of the calcined flint is close to that of cristobalite, while the index of the calcined chalcedony is definitely lower than those of cristobalite and tridymite but agrees very well with that of silica glass. As far as these two properties are concerned, therefore, the calcined chalcedony might be considered as a form of silica glass, while the calcined flint might be either cristobalite or tridymite.

(3) Calcined chalcedony shows an inversion point at about 220° and calcined flint one at about 227°. The cristobalite sample studied was prepared from purified flint by a 7 hours calcination at 1500° with sodium tungstate. It showed an inversion temperature of 244° while the sample of tridymite prepared by a similar calcination for 150 hours at 1300° showed an inversion temperature of 117°.

The inversion temperature of cristobalite has been shown by Fenner to depend upon the history of the sample studied and the values obtained for the calcined chalcedony and flint are therefore not inconsistent with the hypothesis that these materials are cristobalite. Such a hypothesis is evidently supported by the values obtained for the volume change on inversion. The results obtained with the dilatometer, therefore, clearly rule out tridymite and if the calcined chalcedony and flint are to be identified with any of the known forms of silica, cristobalite seems to be the only possibility. But if these materials are to be classed as cristobalite it is evident that density measurements or indices of refraction alone or together are insufficient for the identification of the material, since, especially in the case of the calcined chalcedony, both values are very much lower than any of those recorded for cristobalite.

Since the density and index of refraction of the calcined chalcedony seem to indicate that it should be classed as a kind of silica glass, while the dilatometric data show that it behaves as a kind of cristobalite, some evidence of a more crucial nature seemed necessary before a certain classification of the material could be made. Recourse was therefore had to the determination of the X-ray spectra of the various materials as being the most conclusive evidence that could be secured. Samples of all the materials shown in the table were sent to Dr. A. W. Hull of the General Electric Company who kindly consented to photograph the X-ray spectra. On receipt and comparison of these photographs the following results were obtained. Each spectrum was found to consist of some 25 or more different lines. The spectra of the raw chalcedony, the raw flint, and the quartz were absolutely identical, as were also the spectra of the calcined chalcedony, the calcined flint and the cristobalite This evidence is of such conclusive nature that we are forced to conclude that the calcined flint and chalcedony are composed of cristobalite crystals, while the raw materials are composed of quartz crystals.

How then are we to explain the discrepancy in the density and refractive index data? The following hypothesis seems to be in accord with all of the known facts. Raw chalcedony and flint are colloidal quartz, the individual colloidal crystals being of microscopic or perhaps sub-microscopic size. On calcining the materials the included water is driven out and the individual crystals are transformed *in situ* into cristobalite. The resulting cristobalite crystals would evidently be loosely packed together since in the absence of any flux, there is no opportunity for increase in crystal size. Each cristobalite crystal is thus a colloidal particle made up of a comparatively small number of molecules. The spaces between these colloidal particles of cristobalite must be of sub-microscopic dimensions, so small, in fact, that it is impossible to force a liquid into them. Many of them are possibly completely enclosed. Such a structure would readily account for the very low density displayed by the material. As far as the refractive index is concerned we know that properties such as melting point and vapor pressure depend upon crystal size and it is conceivable that the surface tension forces might also be great enough to distort the crystal form sufficiently to influence the index of refraction. This theory would seem to be borne out by the fact that the calcined chalcedony was not optically homogeneous since different parts of it showed different indices of refraction, the index ranging between 1.456 and 1.470. It is quite possible that many of the crystals are so small that they approach in size the molecules of the liquid phase and the mixture would thus be a colloidal solution of crystals dispersed in their own liquid.

In order to further test the above hypothesis as to the nature of the flint and chalcedony an experiment was made to ascertain whether the density of the calcined chalcedony could be raised by very fine grinding, since if closed pore spaces were present some of these might be disrupted and opened up by this process. A sample of the material was therefore ground in an agate mortar and the finest particles were separated by stirrring up the material with water and allowing it to settle for 48 hours. The milky supernatant liquid was then removed and evaporated to dryness at low temperature. When sufficient material had been obtained in this way, the average size of the particles was determined with a micrometer microscope and they were found to be 0.0023 mm. in diameter. The density of this material was then measured and found to be 2.224, an increase of 2% over the previous value, this showing definitely the presence of extremely small pore spaces.

The facts, that (*a*) the index of the calcined flint agrees with that of cristobalite and (*b*) its density is much higher than that of the calcined chalcedony are readily explained by the presence of the impurities which are of such a nature as to act as a flux at high temperatures and thus to promote the growth of the cristobalite crystals.

[1] *Amer. J. Sci. New Haven*, **36,** 331(1913). See also Ferguson and Merwin, *Ibid.*, **46,** 417 (1917).

THE FREQUENCY-SENSITIVITY OF NORMAL EARS

By H. Fletcher and R. L. Wegel

Research Laboratories of the American Telephone and Telegraph Company and Western Electric Company, Inc.

Communicated by J. J. Carty and F. B. Jewett, November 14, 1921

A large amount of work has been done during the last fifty years in an endeavor to determine in absolute terms the minimum amount of sound that the human ear can perceive. The results obtained by different investigators have varied throughout a very wide range. Two causes contributed to this; namely, that adequate apparatus was not available, and it was not appreciated that so-called normal ears vary so widely in their ability to hear.

The development of the vacuum tube, condenser transmitter, and thermal receiver has given us precision apparatus for work of this kind. In this investigation an air damped receiver was held tightly against the ear by means of a head band. It was actuated by an alternating current which was sent from a vacuum tube oscillator having a range of frequencies from 60 cycles to 6000 cycles per second. By means of a specially constructed attenuator the current entering the receiver could be varied approximately three millionfold. This was accomplished by moving a single dial switch.

By means of condenser transmitters and thermal receivers, this system was calibrated so that from the reading of the attenuator dial switch and the electric current entering it, the alternating pressure impressed upon the ear cavity could be determined in dynes per square centimeter. This calibration was checked by two methods employing entirely different apparatus.

The measurements were made in a room which was especially constructed to eliminate all outside noises. The top, the bottom and the sides of this room were built of a number of alternate layers of loose felt and sheet iron, the final inside layer being felt covered with cheese cloth. It is extremely important that all noise interference be eliminated for this kind of work.

To make a measurement the vacuum tube oscillator was set to give the desired frequency. The attenuator dial was then moved until the sound from the receiver reached the threshold of audibility. Measurements were made upon 93 normal ears. The average of all these measurements is shown in figure 1 by the heavy line. The results of previous observers are also shown on this figure. It was necessary to use the logarithmic scale for pressures due to the large range of values involved. In figure 2 the curves show the variation of the female ears from the average. A similar set of curves for the male ears shows practically the same result. The average of the two sets was the same within the precision of the test. The probable error of observation was less than one-fifth the probable deviation of a single observer's results from the average. The peaks shown are real but vary greatly from one individual to another which make the average curve free from any noticeable peaks.

Audiograms for people of various types of deafness show striking differences in their relative frequency sensitivities. It is expected to make a complete report of work done along this line in the very near future. It is sufficient here to give from our general experience the amount of sound volume in the speech range that is required to make people of various degrees of deafness hear. Persons who have normal hearing require approximately 1/1000 dynes per square centimeter in order to hear sounds in this range. Persons who require a pressure variation of 1/10 dynes per square centimeter are called slightly deaf. Those who require one dyne are partially deaf but can usually follow ordinary conversation. Those who require 10 dynes belong to that class who use ear trumpets or deaf sets to amplify the speech waves. A pressure variation of approximately 1000 dynes can be felt and produces a sensation of pain. It is practicable to assume that people who experience no auditory sensation at these pressures are totally deaf.

This shows that among people who can follow ordinary conversation there is a range in ear sensitivity of more than 1000 and among people who are noticeably deaf there is another range of 1000 making a total range of more than a million for people who can hear or be made to hear by means of amplifying devices.

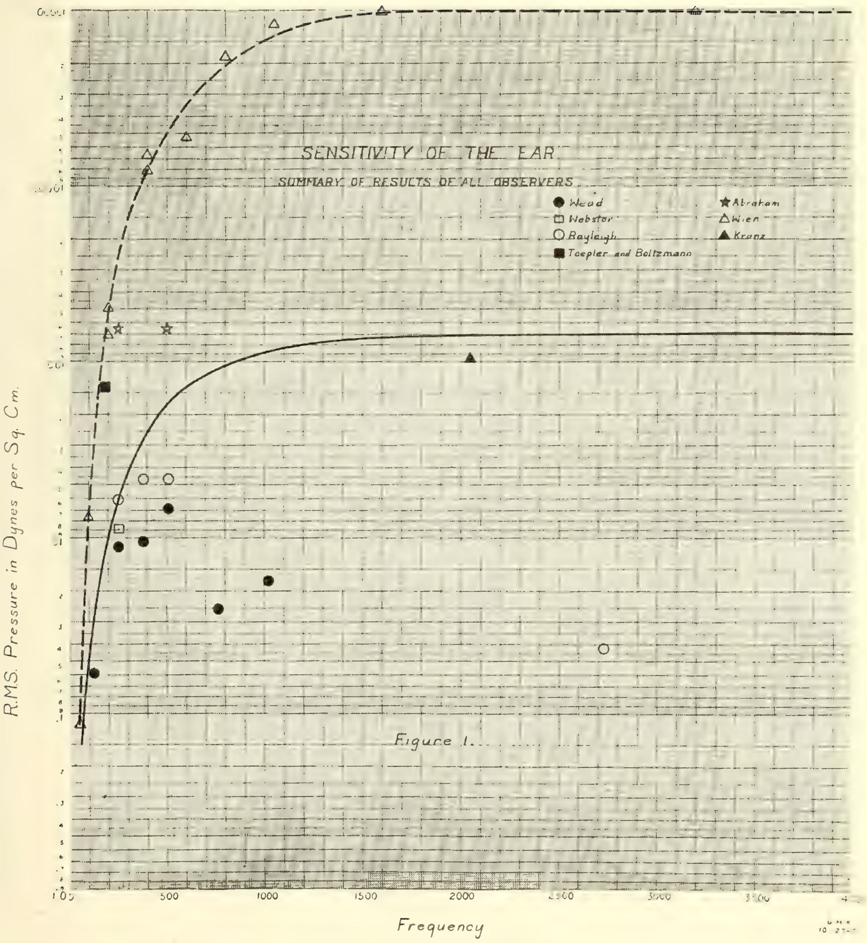
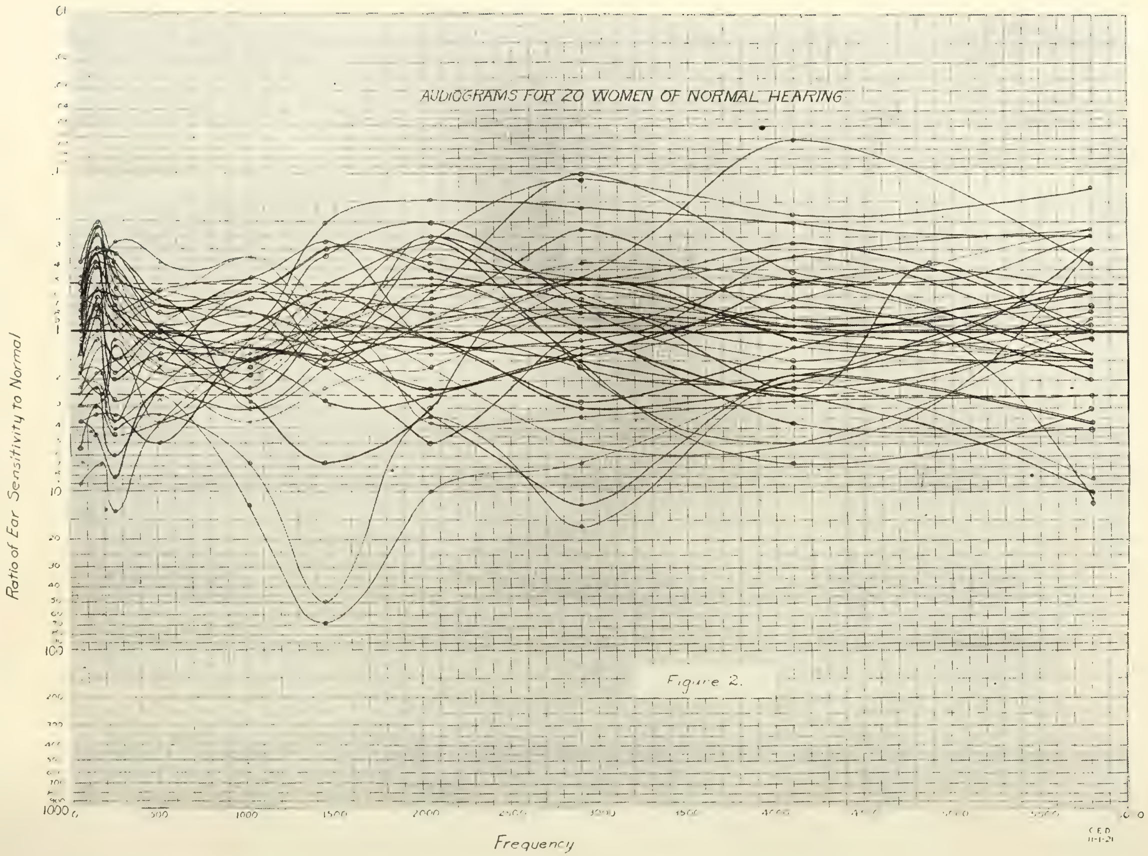

SENSITIVITY OF THE EAR
SUMMARY OF RESULTS OF ALL OBSERVERS
● Wead
☐ Webster
○ Rayleigh
■ Toepler and Boltzmann
★ Abraham
△ Wien
▲ Kranz
R.M.S. Pressure in Dynes per Sq. Cm.
0.0001
500
1000
1500
2000
3000
Frequency

Figure 1.


AUDIOGRAMS FOR 20 WOMEN OF NORMAL HEARING
Ratio of Ear Sensitivity to Normal
Frequency
.01
.1
1
10
20
30
40
50
100
200
300
400
1000
0
500
1000
1500
2000
2500
3000
3500
4000
4500
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6000
C.E.D.
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Figure 2.

CORAL REEFS OF THE LOUISIADE ARCHIPELAGO

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The Louisiade archipelago, consisting of four medium-sized and many small islands east of New Guinea, is well represented on British Admiralty chart 2124 on a scale of about 1:280,000; chart 1477 shows part of the archipelago in greater detail on a scale of about 1:140,000. According to brief accounts by Macgillivray,[1] Thomson,[2] and Maitland,[3] the chief islands are composed of steeply inclined and deeply eroded schists and slates, traversed by quartz veins; they are evidently parts of the mountain range that extends for hundreds of miles along the northern coast of New Guinea, from which they have been separated by strong subsidence after having been eroded to about their present form. The largest island is Tagula, 30 miles in length east-west along the trend of its schists, and 8 or 9 miles in width; it has an embayed shore line and rises in 10 summits to heights of from 1330 to 2645 feet. Near by is the Calvados chain of satellite islands, which begins about 7 miles north of the middle of Tagula and extends 70 miles westward; it includes more than a score of members, the largest having a length of 11 miles and a height of 1110 feet. Tagula and its chain of satellites are enclosed by a superb barrier reef, the irregularly oval circuit of which measures 112 miles in east-west diameter by about 30 miles north-south; it is unquestionably one of the finest reefs of its kind in the whole Pacific.

The smaller islands of Rossel to the east and Deboyne to the northwest of Tagula are also surrounded by sea-level reefs, partly as fringes but mostly barriers. Misima, north of Deboyne, measuring 22 by 10 miles and reaching 3500 feet in height, is peculiar in having no sea-level reefs and in descending rapidly into deep water, altho it is terraced by unconformable reefs at various altitudes. It has therefore suffered a recent uplift after having previously taken part in the subsidence which characterizes the other islands; but its subsidence must have been more rapid than theirs as it has no widely developed barrier-reef lagoon floor, either near present sea level or above or below it.

The Tagula barrier reef and its great lagoon merit special attention from the evidence that they give regarding the verity of certain coral-reef theories. The reef is best developed around the southeastern or windward half of its great oval circuit, where it is interrupted by only 4 passes in a curved distance of 110 miles, and where the reef flat has a width of 2 or 3 miles. The northwestern or leeward half of the barrier is strikingly discontinuous and consists in part of small patches, but more commonly of atoll-like loops and rings, 36 in number, from one to 5 miles in diameter, enclosing little lagoons from 10 to 17 fathoms in depth. The loops and rings of this half of the circuit are separated by as many passages, from $^1/_4$ to 3 miles wide and from 15 to 35 fathoms deep. But the most remarkable features of this part of the barrier are the small or minute but high islands, here to be referred to as outposts, which rise in 22 of the reef loops. The largest of them is only 4 miles in diameter; their heights vary from 40 to 530 feet. Some of them appear to consist of schist, judging by their trends; but according to Maitland some of the others are volcanic and a few are made of limestone. As elements of a barrier reef, these small but high outpost islands are so exceptional as to be almost unique.

The great Tagula lagoon is divided by the Calvados chain of satellite islands into a smaller northern and a larger southern compartment; the northern compartment is of triangular outline, with its base along the dividing chain and its vertex about 10 miles away at the most northern point of the reef; it occupies about $^1/_6$ of the entire reef-enclosed space, which must approach 2000 square miles in total area. The southern compartment measures 20 miles across, and extends east-west along the whole 112 miles of the lagoon length; it occupies about $^4/_6$ of the enclosed area; the remaining $^1/_6$ is taken by Tagula and the satellite islands. The greater part of the lagoon floor in both compartments is a gently undulating plain usually from 25 to 35 fathoms in depth. The depth of the southern compartment increases gradually for a moderate distance from the broad enclosing reef, and more rapidly from the islands of the Calvados chain. The greatest depths, 46 fathoms in the southern or windward compartment and 49 fathoms in the northern or leeward compartment, are in both cases found much nearer the dividing island chain than the outer barrier reef. The exterior slopes of the reef fall off rapidly into deep water; a few soundings show depths of over 600 fathoms two miles from the reef on the west and northwest.

A correct theory of the Louisiade reefs must take account of the great subsidence that the islands have suffered. It would therefore appear that the present sea-level reefs should be regarded as the successors of a long-lived series of upgrowing reefs which have been formed, essentially according to Darwin's theory, by more or less intermittent upgrowth from earlier shore lines of the subsiding mountainous islands. It is probable that where the island slopes were very steep, the reefs, presumably inclining inwards as they grew up, remained attached to the shore as fringes; conversely, where the island slopes were gentler or where low slopes have been broadly submerged, the reefs now form offshore barriers. During the upgrowth of the reefs, some of their detritus must have been swept seaward, to form the submarine talus that descends into deep water; the rest must have been swept into the lagoons, where, reenforced by local organic detritus and probably in smaller measure by detritus from the islands, it appears to have aggraded the "moats" between the reefs and the islands.

It thus seems that the formation of the great undermass of the Louisiade reefs, and especially of the Tagula reef, may well have been consistent with the conditions and processes of Darwin's theory. It should be added that the evidence for the strong subsidence of the Louisiade islands is, in view of their constitution, much more direct than that furnished for the similar subsidence of most reef-encircled volcanic islands in the central Pacific; and that this well certified subsidence of the foundations on which the Louisiade reef-masses have been built up gives immensely greater support for Darwin's theory than is afforded by the atolls of the open Pacific, where the occurrence of subsidence is indicated only by indirect evidence. It remains to inquire whether the Louisiade sea-level reefs, which surmount the great undermass, accord with or contradict other coral-reef theories, especially the newly framed Glacial-control theory of sea-level reefs. This theory was proposed more particularly to account for the atolls and barrier reefs of the supposedly quiescent central Pacific than for the barrier reefs of much disturbed regions like the Louisiade archipelago: nevertheless the Tagula reef in particular affords critical evidence against that theory, as will be made clear by the following considerations.

The Glacial-control theory appears to be based on the conviction that it is the smooth lagoon floors rather than their enclosing reefs which are most in need of explanation, and that the bathymetric relation of the lagoon floors to the level of the ocean reached by the enclosing reefs is normally so nearly constant in all the coral seas that their explanation by Darwin's theory in terms of reef upgrowth and lagoon aggradation on subsiding foundations of irregular form is impossible. A long period of nearly perfect stability of the mid-ocean floor is therefore assumed, altho instability is admitted for islands in the southwestern Pacific; and instead of postulating that lagoon floors represent "moats" that have been heavily aggraded behind the upgrowing reefs during the subsidence of their foundations, a series of ingenious suppositions is invented, of which the chief are: that during Preglacial time many still-standing islands, more or less reef-surrounded, were either worn down to low relief by subaerial erosion or cut down to shallow platforms by marine abrasion; that during the Glacial epochs of the Glacial period the ocean surface was lowered by about 35 fathoms by the withdrawal of water to form continental ice sheets; that the surface waters of the ocean were then so chilled as to kill or greatly weaken reef-building organisms; that islands were then attacked by the waves, which cut low-level benches around them if they were high, or if they were low completely truncated them in platforms at a depth of 35 to 40 fathoms below normal sea level; that as the waters warmed and rose reefs grew up on the margins of the benches and platforms, whereupon the lagoons behind them were moderately aggraded; and that the thickness of the aggrading deposits is greater, and consequently the lagoon depth is less in small than in large lagoons, because the detritus supplied from a linear front-foot of a reef has a smaller interior sector to aggrade in a small lagoon than in a large one. In brief, the long continued stability of reef foundations and the abrasion of sub-lagoon platforms upon them are leading factors of the Glacial-control theory.

It should be noted here that neither the stability of reef foundations nor the abrasion of sub-lagoon platforms is proved by any direct evidence; both of these leading factors are, like the subsidence of atoll foundations in Darwin's theory, assumed because they are thought to be necessary for the explanation of observed facts; and both assumptions are believed to be true because of the apparent success of the explanation that they provide. Hence if it be shown, even in a single instance, that a lagoon floor of typical form and depth has been produced around an island which provides independent evidence contradictory to stability and abrasion and which demands strong subsidence, the fundamental assumptions of the Glacial-control theory will be seriously invalidated.

The bearing of Tagula reef and lagoon on the Glacial-control theory may now be apprehended. Tagula is, as has already been shown, not in a region of long continued and nearly perfect stability, but in one of marked instability; and as will next be shown it has not suffered abrasion by the lowered ocean; yet its lagoon floor is smooth and of a depth accordant with that of other large lagoons in various parts of the Pacific. Hence long continued stability and extensive low-level abrasion are not essential factors in the production of this fine example of a barrier-reef lagoon floor. But if these factors are not essential in Tagula, they should not be regarded as essential anywhere else; and their adoption as the leading postulates of the Glacial-control theory is therefore unnecessary; flatness of lagoon floors and their accordant depths may be explaned elsewhere as well as in Tagula as the result of long continued aggradation on subsiding foundations of uneven surface.

The evidence that Tagula has not suffered abrasion by the low Glacial ocean, and hence that the reef-building organisms around Tagula were not seriously weakened by the lowered temperatures of the lowered ocean in the Glacial epochs, is found in partly in the absence of charted cliffs on the shores of the main island where the barrier reef becomes a fringe, partly in the absence of similar cliffs on the exposed sides of the satellite islands at either end of the Calvados chain where it approaches the barrier reef, and partly in the presence of the outpost islands in the barrier-reef loops around the northern lagoon compartment.

As to the first line of evidence based on the absence of cliffs on Tagula: If abrasion by the lowered ocean had operated long enough to cut a platform 10 or 20 miles wide beneath the present floor of the southern compartment, it ought at the same time to have cut spur-end cliffs on the north shore of the main island, where the defending reef is a fringe only half a mile wide; and these cliffs ought still to show the upper part of their faces as plunging cliffs, now that the ocean has resumed its normal level; but the charts show no such cliffs.

The second line of evidence based on the absence of cliffs on the Calvados islands is similarly argued. It may be added that the absence of cliffs at these significant points on the charts of the Louisiade islands does not appear to be due to poor charting; for on the coast of Misima, where Maitland observed the white limestone scarps of elevated reefs, the charts clearly show a shore cliff, and a legend is printed along it: "Cliffs 100 feet high."

As to the third line of evidence: The little outpost islands are so numerous in the Tagula barrier-reef loops around the northern lagoon compartment and around the western part of the southern compartment that it seems unreasonable to believe the waves of the lowered Glacial ocean could have cut their way behind the outposts efficiently enough to abrade a platform 10 miles in width. Not only so, the outpost islands show no sign of being clift on their outer sides. One of them, Utian, a miles across and 480 feet high, is reported by Maitland to consist of volcanic rocks; but it is not a young volcanic cone built up in Postglacial time, for the chart shows it to have well dissected form, with three slender points enclosing two small bays turned toward the outer ocean; yet the points are not cut back in plunging cliffs. Another outpost not far away is said by the same observer to consist of limestone; this island cannot have been made and elevated since an assumed platform was abraded, for the height of the island, 530 feet, is so great that, in such case, the platform thereabouts ought to be more or less emerged; and it cannot have been made and raised before the platform was cut, for in such case the limestone ought to have been consumed by the waves that cut the platform.

The small outpost islands of the Tagula barrier reef therefore give strong confirmation of the evidence against abrasion derived from the absence of plunging cliffs on the north side of the main islands and on the terminal members of the Calvados chain. But if the northern compartment of the Tagula lagoon, which is alone as large as many an atoll, is thus shown not to be underlaid by an abraded platform, there is no sufficient reason for thinking that the southern compartment or indeed any other barrier reef or atoll lagoon in the whole Pacific has any such smoothly prepared foundation. Surely, if the flatness of the floor and its normal depth in both compartments of the Tagula lagoon have been brought about in a region of instability and without the aid of abrasion in furnishing a smooth sub-lagoon platform, there is no sufficient reason for assuming that other flat lagoon floors of ordinary depth can have been prepared only on smooth platforms abraded at a standard depth across still-standing islands. It is possible that the Glacial lowering of the ocean surface by a moderate amount may have contributed in a manner that I have suggested elsewhere,[4] to the production of many lagoon floors 30 or 40 fathoms in depth; but Glacial changes of ocean level do not seem otherwise to have left recognizable marks of their occurrence in the Louisiade archipelago. Crustal deformation has been dominant; and the great changes of shore lines thus determined appear to have been merely played upon by the inferred oscillations of ocean level during the Glacial period.

This discussion is believed to show that, apart from such changes of ocean level as are inherently probable altho they are not well known either in amount or in effects, the assumptions of the Glacial-control theory are not applicable in the production of Tagula reef and lagoon floor; and hence we may fairly conclude that these assumptions are not essential to the production of similar reefs and lagoon floors elsewhere. This argument, in which the evidence furnished by one outspoken witness for Darwin's theory and against the Glacial-control theory is given wide application, would not be valid if other witnesses were equally outspoken elsewhere against Darwin's theory and for the Glacial-control theory; but such is not the case. It must be remembered that the two main postulates of the Glacial-control theory, namely, long continued stability of reef foundations in the mid-Pacific and the abrasion of sub-lagoon platforms by the lowered Glacial ocean, are not based on direct evidence but are assumed because they are supposed to be necessary for the explanation of smooth lagoon floors of standard depths. Not a single example of an abraded platform has been discovered under recently uplifted reefs; and a good number of mid-Pacific islands which have a decipherable recent history are found not to have been long stable but to have suffered various changes of level. In other words, where other outspoken witnesses are found, their testimony is, like that of Tagula, for Darwin's theory of upgrowing reefs on subsiding foundations of whatever shape. A good number of examples of this kind could be adduced if space permitted.

But although the inhibition of reef growth and the resulting abrasion of low-level platforms by the Glacial ocean thus appear to be excluded from the greater part of the coral seas, it is highly probable that reef-building organisms may have been weakened or killed and that abrasion of platforms may have taken place around islands near the margin of the coral seas; and at least some of those islands ought now to show plunging cliffs in evidence of their possession of submerged platforms; but even there the islands need not have been stable. This aspect of the coral-reef problem is examined in an essay submitted to the Geological Society of America for publication in its Bulletin.

The foregoing discussion will be presented in greater detail in the Annals of the Association of American Geographers.

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[1] J. Macgillivray, *Narrative of the Voyage of H. M. S. Rattlesnake*, London, 1852. 2 vols. See i, 182; ii, 72.

[2] B. H. Thomson, "New Guinea: Narrative of an Exploring Expedition to the Louisiade and D'Entrecasteaux Islands, "*Proc. Roy. Geogr. Soc.*, **11,** 1889 (525–542).

[3] A. G. Maitland, "Geological Observations in British New Guinea," *Queensland Geol. Surv. Pub.*, **85,** 1892. "Salient Geological Features of New Guinea," *Journ. W. Austral. N. H. Soc.*, **2,** 1905 (32–56).

[4] "Problems Associated with the Study of Coral Reef," *Sci. Monthly*, **2,** 1916 (565).

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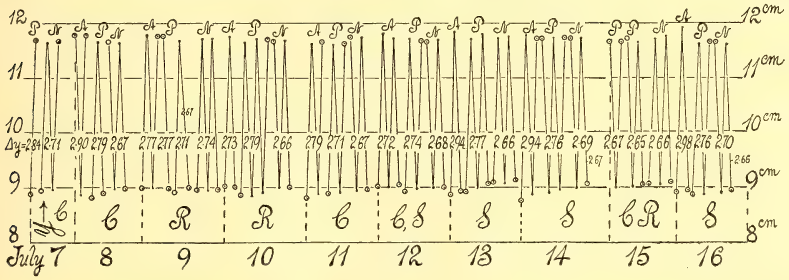
*EXPERIMENTS WITH THE VACUUM GRAVITATION NEEDLE**

By Carl Barus

Department of Physics, Brown University

Communicated November 11, 1921

1. *Apparatus.*—At the end of the previous summer's experiments, a form of apparatus was installed capable of exhaustion and containing a gravitation needle made of wires as thin as possible (bronze, .24 mm. in diameter) compatibly with the lead weights, m, at the ends. The object of this was to diminish the radiation effect to an inferior limit, always remembering that the lateral area of the shots m is necessarily in presence. The narrow case was a rectangle of brass with reentrant sides, into which plates of thick glass could be sealed with cement free from air leakage.



Each attracting weight M had a mounting quite independent of the case and could easily be moved between stops from one side to the other of it by a crank-like arrangement.

The half silver method of reading the deflections of the needle and other details were retained, with the scale distance, $L = 447.5$ cm.

The masses of the attracted lead balls at the end of the filamentary needle were $m = .6295$ grams each, though the quartz fibre would have been strong enough to hold a larger mass to advantage. The distances R of the center of the mass M from the center of the shot m were obtained by calipering.

The moment of inertia of the needle $2l=22.0$ cm. long between centers of $m=.6295$ g has for its main part $2ml^2=152.5$. To this is to be added the moment of inertia of the stem wire ($\rho=.0044$ grams per cm.), 3.9; of the oblique wire brace or tie, 1.0; and of the glass filament added for ridgity, 2.0; making a total of $N=159.4=ml^2(1+s)=152.5(1+.0452)$.

The torsion coefficient of the wire was found from the period T_1 of the needle vibrating in vacuo, by a stop watch. As T is over 5 minutes, it is necessary that the arc of vibration be relatively large; otherwise the passage through equilibrium is too slow. Periods were found as follows: $T_1=311.5$, 310.0, 312.0, 311.5, 311.0, 311.2; Mean $T_1=311.2$ ±.28 sec. The discrepancies result from the presence of radiation forces.

The logarithmic decrement, λ, obtained from observations of consecutive elongations showed $\epsilon^\lambda=1.35$ in the first and 1.34 in the (better) second series, which makes $\lambda=.293$ nearly. Hence the true period T of the needle is

$$T=T_1/\sqrt{1+\lambda^2/4\pi^2}=T_1/1.0011.$$

The intervals on the stop watch moreover were corrected (on comparison with a chronometer) by the factor 1.0055. This with the preceding equation gives

$$T=T_1(1.0012)=312.6 \text{ sec.}$$

2. *Equations.*—The approximate equation for the gravitation constant γ, containing all quantities to be measured when the needle of semi-length l is used to find the torsion coefficient of the quartz fibre, is

$$\gamma'=(\pi^2NR^2/LT^2lMm)\,\Delta y \tag{1}$$

Δy being the ultimate double amplitude for the scale distance L, when the attracting weight M passes from side to side of m.

Since the stem is also appreciably attracted, a correction must be made for it. If t/τ is the ratio of torques for stem and mass m separately

$$t/\tau=(\rho R/lm)(\sqrt{R^2+l^2}-R) \tag{2}$$

when ρ is the mass per cm. of the stem.

Hence finally the corrected constant is

$$\gamma=\gamma'/(1+t/\tau). \tag{3}$$

For the case of two attracting masses M' and M'', one at each end of the needle and coöperating, we should have

$\gamma=K'\Delta y'=K''\Delta y''=(\Delta y'+\Delta y'')/(1/K'+1/K'')$; and $\Delta y=\Delta y'+\Delta y''$, whence

$$\gamma=\Delta y/(1/K'+1/K'') \tag{4}$$

K' and K'', the corrected coefficients, are usually not very different in value.

If in equation (1) we replace N by $2ml^2(1+s)$ where $s=.0453$, m vanishes from the equation (except in the corrections) which now takes the simpler form

$$\gamma = 2\pi l R^2(1+s-t/\tau)\,\Delta y/LMT^2 \qquad (5)$$

showing that l virtually enters in the first power only.

3. *Observations.—Large attracting masses M coöperating.* The constants entering equation (1) have been given for the needle. The new data are

$$M = 3368 \text{ grams} \qquad R = 5.67 \text{ cm.}$$

Hence the following relations result:

$$\gamma' = 10^{-8}\times 4.960\,\Delta y'; \text{ and } t/\tau = .0241,$$

which in equation (3) gives

$$\gamma = 10^{-8}\times 4.844\ \Delta y'.$$

The second mass was a little smaller; viz., $M'' = 2947$ grams and $R = 5.43$ cm.

Hence the approximate value of K'' is $10^{-8}\times 5.198$. The value of t/τ is .0236 so that in view of the factor 1/1.0236

$$\gamma = K''\Delta y'' = 10^{-8}\times 5.078\,\Delta y''.$$

If the balls, M, act together

$$\gamma = \Delta y(1/K' + 1/K'') = 10^{-8}\times 2.479\,\Delta y$$

which is partically the fourth part of $K' + K''$. The normal deflection should be $\gamma/K = 2.69$ cm.

The observations, begun on July 7 and continued for some time, 3 times daily, forenoon (A), afternoon (P) and night (N), with periods of from 30 minutes to one hour between consecutive observations,[†] are given in the figure, the mean value of Δy being inscribed on each group of observations terminating in little circles. As the normal value should be $\Delta y = 2.69$ cm., it is obvious that only the night (N) observations are acceptable. On July 7 the apparatus, which had just been set up, was slowly approaching thermal equilibrium.

If we exclude the first night datum as the apparatus was too fresh and the exceptional datum during the fog of July 9, and take the last doublets specified, the mean of the eight deflections is $\Delta y = 2.666 \pm .0017$ cm. Therefore

$$\gamma = 10^{-8}\times 2.479\,\Delta y = 10^{-8}\times 6.609$$

with a mean error in T and Δy together, of less than .3%. This is as near the normal value as one may hope to get, seeing that the small individual deflections, Δy, must be read off on a millimeter scale. If all night observations were taken $\gamma = 6.63$ would result.

Night observations made on an exhausted gravitational apparatus, therefore, give promise of trustworthy results, particularly if a season of uniform temperature is chosen and a relatively finer quartz fibre is inserted. While the exhaustion here in question has not been carried further than a few millimeters, the experiments are all aiming at the final test under complete exhaustion.

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[*] Advance note from a report to the Carnegie Institution of Washington.

[†] To secure better vision the scale was slightly shifted on July 13.

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CARL HARTMAN, University of Texas, $500. For the study of the oestrous cycle of the opossum.

WILLIAM BOWIE, U. S. Coast and Geodetic Survey, $250. For the work of the Ukiah Latitude Station.

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STRUCTURAL CHANGES ASSOCIATED WITH FACTOR MUTATIONS AND WITH CHROMOSOME MUTATIONS IN DATURA[1]

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The fact has recently been emphasized[2] that two distinct types of mutation may occur in plants—those which are due to the change of a single factor or gene and those which are due to the addition of one or more entire chromosomes. The Jimson Weed, *Datura Stramonium*, displays both of these mutation types. If we consider the purple-flowered, spiny fruited, many-noded form the primitive type, three "factor" mutations have occurred, giving rise to white-flowered, smooth-fruited, and few-noded types; and twelve distinct "chromosome" mutations have been identified and named, each of which has been found to be associated with the addition of an extra chromosome to a particular one of the twelve pairs. Tetraploid plants and other chromosomal aberrations have also been found.

These mutations have been identified by various external characters. The purpose of the present investigation (of which this paper is a preliminary report) is to study and compare the *structure* of these mutant forms, both as to gross external morphology and as to internal anatomy; and thus to determine the structural effects produced by a single factor and those produced by a single entire chromosome. In this way it may be possible to begin an analysis of the factorial constitution of each of the chromosomes.

Externally, various measurements of the gross structure of the plant body were made. Branching in the Jimson Weed is essentially dichotomous and the two branches are usually of decidedly different lengths. These measurements were of (1) the trunk length (height of stalk to the first fork; (2) the width of the angle formed by the branches at the first fork of the stem; (3) the width of the angles formed by the branches at the two second forks; (4) the width of the angles formed by the branches at the four third forks; (5) the length of the internodes forming the first fork; (6) the lengths of the two pairs of internodes forming the two second forks; (7) the lengths of ten of the younger internodes (all of the same age) near the ends of the branches; (8) the diameter of these ten internodes; and (9) the ratio between the length of the two branches above each of these ten internodes. In each type studied, a considerable number of plants were measured and the measurements averaged.

The various factor mutations were compared with the normal type (purple-flowered, spiny-fruited and many-noded). The white-flowered mutant was not found to differ appreciably from the normal type. The smooth-fruited form, however, has significantly narrower angles, shorter internodes, greater stem diameter and greater difference between the two branches at a fork than does the spiny-fruited one; and the few-noded type has a significantly shorter trunk, shorter internodes, narrower stem diameter and greater difference between the two branches at a fork than does the many-noded form. These constitute excellent examples of the manifold effects of a single factor upon the plant body as a whole.

Internally, few constant and characteristic differences between these mutant types and the normal form have as yet been distinguished, but in view of the notable differences in external structure it seems likely that a more complete and detailed anatomical study may reveal such differences.

An investigation was made of the gross external structure of several of the chromosome mutants with a single extra chromosome and each was found to display characteristic and significant differences from the normal type. Compared with the normal, "Reduced," for instance, shows a significant decrease in trunk length, angle width and stem diameter; "Cockleburr" an increase in angle width and a decrease in stem diameter and in internode length; "Globe" a decrease in internode length and an increase in stem diameter and in the difference in length between the two branches at a fork, and "Buckling" an increase in angle width.

Furthermore, a number of these chromosome mutants differ radically from the normal type in their internal structure. Both "Poinsettia" and "Wiry," for example, tend to have an asymmetrical central cylinder, and opposite the thin portions of the cylinder the cortex breaks open and produces a characteristic lesion. In these mutants wedges of wood also extend for some distance into the pith. In both "Cockleburr" and "Wedge," vessels are much more abundant than in the normal type and show a great variation in size. "Reduced" has a well-developed ring of sclerenchyma at the outer edge of the pith and the inner edge of the cortex. "Globe" tends to have thick-walled vessels, very round in cross section.

In "Buckling" the woody cells are very weakly lignified. Other distinctions will doubtless reveal themselves on a more intensive study.

Tetraploid plants from two distinct races were examined. Both showed an elongation of the trunk and an increase in the width of the first angle, the first pair of internodes and the stem diameter. Internally, the cells were found to be significantly larger than in the normal type and the woody tissue tended to be less strongly lignified.

Only a preliminary survey of the field has as yet been made. The authors believe, however, that these Datura cultures provide exceptionally promising material for a study of the effect of specific factors and of specific chromosomes, particularly upon structural characters; and they hope through further investigation to be able to contribute materially to a factorial analysis of the chromosomes of this species.

[1] Paper presented before the Botanical Society of America, December 28, 1921.

[2] Albert F. Blakeslee, "Types of Mutations and Their Possible Significance in Evolution," *Amer. Naturalist,* **55,** 1921 (254–267); and "Variations in Datura Due to Changes in Chromosome Number," *Ibid.*, Jan.–July, 1922.

THE RIEMANN GEOMETRY AND ITS GENERALIZATION

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1. One of the simplest ways of generalizing Euclidean Geometry is to start by assuming (1) that the space to be considered is an n-dimensional manifold in the sense of Analysis Situs, and (2) that in this space there exists a system of curves called *paths* which, like the straight lines in a euclidean space, serve as a means of finding one's way about.

These paths are defined as the solutions of a system of differential equations,

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{1.1}$$

in which the Γ^i_{jk}'s are analytic functions of $(x^1, x^2, \ldots, x^n)$ and the indices i, j, k run from 1 to n. The second term is a summation with regard to j and k in accordance with the usual convention in such formulas that any term represents a summation with respect to each letter which appears in it both as a subscript and as a superscript.

Since the second term is a quadratic form in $\frac{dx^i}{ds}$, there is no loss of generality in assuming, as we do, that

$$\Gamma^i_{jk} = \Gamma^i_{kj} \tag{1.2}$$

This definition of the *paths* is immediately suggested by the fact that the differential equations of the straight lines in a euclidean space which are

$$\frac{d^2x^i}{ds^2}=0 \tag{1.3}$$

in cartesian coördinates, take the form (1.1) in general coördinates, the Γ's now being such that there shall exist an analytic transformation of $(x_1, x_2, \ldots . x_n)$ converting (1.1) into (1.3).

2. This generalized geometry has been studied by H. Weyl in his book, "Raum, Zeit, Materie," Berlin, 1919, and in Vol. 1 of the "Mathematische Zeitschrift." It has also been considered by A. S. Eddington in *Proc. Roy. Soc. London,* **99A** (1921). Both these authors define it in terms of a generalization of Levi-Civita's concept of infinitesimal parallelism rather than by the more natural idea of a system of paths.

It reduces to the Riemann geometry if we assume that there exists a quadratic form

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \qquad (g_{ij}=g_{ji}) \tag{2.1}$$

with respect to which the paths are geodesics, i.e., if the curves for which the integral

$$\int\sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}\,ds \tag{2.2}$$

is stationary satisfy the differential equations (1.1). The conditions that (2.2) be stationary are

$$g_{\alpha i}\Gamma^{\alpha}_{jk} = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{ki}}{\partial x_j}-\frac{\partial g_{jk}}{\partial x^i}\right). \tag{2.3}$$

When solved for the derivatives of the g's these conditions become

$$\frac{\partial g_{ij}}{\partial x^k} - g_{\alpha j}\Gamma^{\alpha}_{ik} - g_{i\alpha}\Gamma^{\alpha}_{jk} = 0. \tag{2.4}$$

3. In general there exists no quadratic form (2.1) for which the paths are geodesics. For example, in a two-dimensional manifold the system of paths defined by

$$\begin{aligned}\frac{d^2x^1}{ds^2}+(x^1-x^2)\left(\frac{dx^1}{ds}\right)^2&=0,\\ \frac{d^2x^2}{ds^2}+(x^1-x^2)\left(\frac{dx^2}{ds}\right)^2&=0\end{aligned} \tag{3.1}$$

is one for which the equations (2.4) are inconsistent. Therefore there exists no quadratic form (2.1) of which the paths (3.1) are geodesics.

The problem of determining under what conditions the geometry of paths is Riemannian is one of the inverse problems of the Calculus of Variations. But it does not seem hitherto to have been solved. Before considering it further we must mention a few general theorems, which have already been noted more or less explicitly by Weyl and Eddington.

4. If we put $x^i = \varphi^i(\bar{x}^1, \ldots . \bar{x}^n)$, thus introducing a new set of coördinates, the equations (1.1) become

$$\frac{d^2\bar{x}^i}{ds^2} + \bar{\Gamma}^i_{jk} \frac{d\bar{x}^j}{ds} \frac{d\bar{x}^k}{ds} = 0, \tag{4.1}$$

where

$$\frac{\partial^2 x^p}{\partial\bar{x}^i \partial\bar{x}^j} + \Gamma^p_{qr} \frac{\partial x^q}{\partial\bar{x}^i} \frac{\partial x^r}{\partial\bar{x}^j} = \bar{\Gamma}^t_{ij} \frac{\partial x^p}{\partial\bar{x}^t} . \tag{4.2}$$

Expressing the conditions of integrability of these equations regarded as differential equations for determining the x's as functions of the $\bar{x}$'s, when the Γ's and $\bar{\Gamma}$'s are known, we obtain

$$\frac{\partial x^q}{\partial\bar{x}^i} \frac{\partial x^r}{\partial\bar{x}^j} \frac{\partial x^s}{\partial\bar{x}^k} B^p_{qrs} = \frac{\partial x^p}{\partial\bar{x}^t} \bar{B}^t_{ijk} \tag{4.3}$$

where

$$B^p_{qrs} = \frac{\partial\Gamma^p_{qs}}{\partial x^r} - \frac{\partial\Gamma^p_{qr}}{\partial x^s} + \Gamma^p_{\alpha r} \Gamma^\alpha_{qs} - \Gamma^p_{\alpha s} \Gamma^\alpha_{qr}. \tag{4.4}$$

Equation (4.3) may also be written

$$\frac{\partial x^q}{\partial\bar{x}^i} \frac{\partial x^r}{\partial\bar{x}^j} \frac{\partial x^s}{\partial\bar{x}^k} \frac{\partial\bar{x}^t}{\partial x^p} B^p_{qrs} = \bar{B}^t_{ijk}. \tag{4.5}$$

This states that the functions $\bar{B}$ are the components in the coördinate system $\bar{x}$ of a tensor, B, which is contravariant of the first order and covariant of the third order, with respect to the group of all transformations of the paths into themselves. This tensor is called the *curvature tensor* of the manifold.

From (4.4) it follows that

$$B^p_{qrs} + B^p_{qsr} = 0, \tag{4.6}$$

and

$$B^p_{qrs} + B^p_{rsq} + B^p_{sqr} = 0. \tag{4.7}$$

The theory of covariant differentiation (cf. Ricci and Levi-Civita in *Math. Ann.*, **54** (1901) can be generalized at once to the geometry of paths by replacing the Christoffel symbols $\left\{ {jk \atop i} \right\}$ by the functions Γ^i_{jk} in all formulas. In particular it is easily proved by means of (4.1) and (4.2) that the operation of covariant differentiation converts any tensor into a tensor of higher order. The formulas for covariant differentiation of sums and products of tensors also generalize without change, and also the theorem that if a_{ij} is any tensor and a_{ijkl} is its second covariant derivative,

$$a_{ijkl} - a_{ijlk} = a_{\alpha j} B^\alpha_{ikl} + a_{i\alpha} B^\alpha_{jkl}. \tag{4.8}$$

5. Returning now to the question as to the conditions on the Γ's that they shall yield a Riemann geometry, we observe that the left hand member of (2.4) is the covariant derivative of g_{ij}. Thus (2.4) may be written

$$g_{ijk} = 0. \tag{5.1}$$

By (4.8)

$$g_{ijkl} - g_{ijlk} = g_{\alpha j} B^{\alpha}_{ikl} + g_{i\alpha} B^{\alpha}_{jkl} \tag{5.2}$$

which combined with (5.1) gives

$$g_{\alpha j} B^{\alpha}_{ikl} + g_{i\alpha} B^{\alpha}_{jkl} = 0. \tag{5.3}$$

If these equations be differentiated covariantly, we get

$$g_{i\alpha} B^{\alpha}_{jklm} + g_{\alpha j} B^{\alpha}_{iklm} + g_{i\alpha m} B^{\alpha}_{jkl} + g_{\alpha jm} B^{\alpha}_{ikl} = 0. \tag{5.4}$$

Hence by (5.1)

$$g_{i\alpha} B^{\alpha}_{jklm} + g_{\alpha j} B^{\alpha}_{iklm} = 0. \tag{5.5}$$

Proceeding in this manner we get a sequence of equations, namely

$$\begin{gathered} g_{i\alpha} B^{\alpha}_{jklmn} + g_{\alpha j} B^{\alpha}_{iklmn} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_{i\alpha} B^{\alpha}_{jklmn \cdots p} + g_{\alpha j} B^{\alpha}_{iklmn \cdots p} = 0. \end{gathered} \tag{5.6}$$

If the geometry of paths is Riemannian (5.1) must be satisfied and hence also (5.3), (5.5) and (5.6). The equations (5.3), (5.5) and (5.6) are linear equations in the $(n + 1)n/2$ functions g_{ij} with coefficients which are functions of the Γ's alone. Hence the algebraic conditions for the consistency of (5.3), (5.5) and (5.6) regarded as linear equations in the g's are necessary conditions on the Γ's that the geometry of paths shall be Riemannian.

6. Now suppose that equations (5.3) and (5.5) are algebraically consistent in the g's, and that the rank of the matrix of the B's is such that the g's are determined by (5.3) to within a factor, which is at most a function of the x's. Let g_{ij} stand for a particular solution of (5.3) and (5.5). If (5.3) be differential covariantly with respect to x_m, we have in consequence of (5.5)

$$g_{i\alpha m} B^{\alpha}_{jkl} + g_{\alpha jm} B^{\alpha}_{ikl} = 0. \tag{6.1}$$

Since these equations are of the same form as (5.3), it follows from the above hypothesis about (5.3) that

$$g_{ijk} = \varphi_k g_{ij}, \tag{6.2}$$

where φ_k is a covariant vector. Substituting these expressions in

$$g_{ijkl} - g_{ijlk} = 0, \tag{6.3}$$

which follows from (5.2) and (5.3), we obtain

$$\frac{\partial \varphi_l}{\partial x^k} = \frac{\partial \varphi_k}{\partial x^l}.$$

Hence the vector φ_k is the gradient of a scalar function, and can be put in the form

$$\varphi_k = -\frac{\partial \log \lambda}{\partial x^k}. \qquad (6.4)$$

Substituting this value of φ_k in the right member of (6.2) and the explicit expression for g_{ijk} (cf. 2.4) in the left member, we obtain

$$\frac{\partial}{\partial x^k}(\lambda g_{ij}) - \lambda g_{pi}\, \Gamma^p_{jk} - \lambda g_{pj}\, \Gamma^p_{ik} = 0. \qquad (6.5)$$

Consequently the functions λg_{ij} satisfy (2.4) and give a Riemann geometry.

It is interesting to note that the geometry of paths obtained from (6.2) without imposing the condition (6.4) is the geometry used by Weyl as the basis for a combined electromagnetic and gravitational theory. For (6.2) is equivalent to

$$g_{\alpha k}\, \Gamma^\alpha_{ij} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) + \frac{1}{2}(g_{ik}\, \varphi_j + g_{jk}\, \varphi_i - g_{ij}\, \varphi_k). \qquad (6.6)$$

7. Let us now assume that (5.3) are algebraically consistent, and that all of their solutions satisfy (5.5). Let $g^{(1)}_{ij}, g^{(2)}_{ij} \ldots\ldots g^{(p)}_{ij}$ be a complete set of solutions. The general solution is expressible in the form

$$g_{ij} = \varphi^{(1)}\, g^{(1)}_{ij} + \varphi^{(2)}\, g^{(2)}_{ij} + \ldots\ldots + \varphi^{(p)} g^{(p)}_{ij}. \qquad (7.1)$$

Differentiating (5.3) covariantly, we get (6.1), and consequently we must have

$$g^{(\alpha)}_{ijk} = \lambda^{(\alpha 1)}_k\, g^{(1)}_{ij} + \ldots\ldots + \lambda^{(\alpha p)}_k\, g^{(p)}_{ij}. \qquad (7.2)$$

The p^2 vectors $\lambda^{(\alpha\beta)}_k$ $(\alpha,\ \beta = 1, \ldots . p)$ must be such that the functions $g^{(\alpha)}_{ijk}$ satisfy (6.3). On substituting (7.2) in (6.3) we find

$$\frac{\partial \lambda_k^{(\alpha\beta)}}{\partial x^l} - \frac{\partial \lambda_l^{(\alpha\beta)}}{\partial x^k} + \sum_{\gamma=1}^{p}\left(\lambda_k{}^{(\alpha\gamma)}\, \lambda_l{}^{(\gamma\beta)} - \lambda_l{}^{(\alpha\gamma)} \lambda_k{}^{(\gamma\beta)}\right) = 0. \qquad (7.3)$$

When we express the condition that the functions g_{ij} given by (7.1) shall satisfy the conditions $g_{ijk} = 0$, we find in consequence of (7.2) that the functions $\varphi^{(\alpha)}$ must satisfy the equations

$$\frac{\partial \varphi^{(\alpha)}}{\partial x^k} + \sum_{\beta=1}^{p} \varphi^{(\beta)}\, \lambda_k{}^{(\beta\alpha)} = 0. \qquad (7.4)$$

In consequence of (7.3) this system of equations is completely integrable, and hence there exists a set of φ's which by means of (7.1) determine a system of g's which yield a Riemann geometry. Hence we have the theorem: *In order that the geometry of paths shall be a Riemann geometry it is sufficient that the* Γ's *be such that the equations (5.3) be algebraically consistent, and that all of their solutions satisfy (5.5).*

RICCI'S PRINCIPAL DIRECTIONS FOR A RIEMANN SPACE AND THE EINSTEIN THEORY

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In 1904 Ricci (*Atti R. Ist. Veneto*, **62**, 1230) developed the idea of principal directions in a Riemann space of n-dimensions, and in doing so introduced the contracted curvature tensor, which is fundamental in the Einstein theory, and gave a geometrical interpretation to it. A space in which these principal directions are completely indeterminate may be thought of as possessing a homogeneous character. We derive Ricci's results by a slightly different method, and then show that the three types of space, chosen by Einstein in 1914, 1917 and 1919, as spaces free from matter are of this homogeneous character, and include all types of such spaces.

Consider a Riemann space V_n of n-dimensions with the linear element

$$ds^2 = g_{ij}\, dx^i\, dx^j \; (g_{ij} = g_{ji}). \tag{1}$$

The right-hand member represents the sum of terms as i and j take on the values $1, \ldots\ldots n$, in accordance with the usual convention in such formulas that any term represents a summation with respect to each letter which appears in it both as a subscript and a superscript.

Suppose that we have in V_n n orthogonal unit vectors and let λ^i_h $(i = 1, \ldots\ldots n)$ denote the contravariant components of the vector (h). Then we have

$$g_{ij}\lambda^i_h\lambda^j_k = \delta_{hk}, \tag{2}$$

where

$$\delta_{hk} = \begin{matrix} 1 \text{ for } h = k \\ 0 \text{ for } h \neq k. \end{matrix} \tag{3}$$

If we take the surface consisting of the geodesics tangent at a point P to the pencil of directions determined by the lines of two congruences (h) and (k) through P, the gaussian curvature of this surface at P is given by

$$r_{hk} = R_{pq,rs}\, \lambda^p_h\, \lambda^q_k\, \lambda^r_h\, \lambda^s_k, \tag{4}$$

where $R_{pq,rs}$ is the Riemann tensor of the first kind. By definition r_{hk} is the Riemann curvature of V_n at P for the directions (h) and (k) (cf. Bianchi, **1**, 342).

Since the n vectors are mutually orthogonal, we have

$$\sum_k \lambda^q_k\lambda^s_k = g^{qs}, \tag{5}$$

where g^{qs} is the cofactor of g_{qs} in the determinant of the g's divided by the determinant. Hence from (4) we have

$$\sum_k r_{hk} = - R_{pr}\lambda_h^p\lambda_h^r, \tag{6}$$

where

$$R_{pr} = - g^{qs}R_{pq,rs}, \tag{7}$$

and consequently is the contracted curvature tensor. From (4) it follows that $r_{hh} = o$, and therefore

$$\rho_h = \sum_k r_{hk} = - R_{pr}\lambda_h^p\lambda_h^r \tag{8}$$

is the sum of the Riemann curvatures determined by (h) and each of $n-1$ directions orthogonal to (h). Ricci calls ρ_h the *mean curvature* of V_n for the direction (h) at the point. Thus Ricci not only obtained the fundamental contracted tensor in 1904, but gave a geometrical interpretation of it.

In general as the vector (h) is changed the value of ρ_h varies. Since the components λ_h^p are bound by the equations $g_{pr}\lambda_h^p\lambda_h^r = 1$ in order to find the directions giving the maximum and minimum values of ρ_h, we equate to zero the derivatives with respect to λ_h^r $(r = 1, \ldots\ldots n)$ of

$$\rho_h = - \frac{R_{pr}\lambda_h^p\lambda_h^r}{g_{pr}\lambda_h^p\lambda_h^r}. \tag{9}$$

This gives

$$(R_{pr} + \rho_h g_{pr})\lambda_h^p = 0. \tag{10}$$

Hence the maximum and minimum values of ρ_h are the roots of the equation

$$\left| R_{pr} + \rho g_{pr} \right| = 0, \tag{11}$$

and the direction for each ρ is given by the corresponding n equations (10) for $r = 1, \ldots\ldots n$. Following Ricci we call these n directions the *principal directions* of the V_n. It is readily shown that if the roots of (11) are distinct, the n corresponding directions are mutually orthogonal.

If $\lambda_{h|p}$ denote the covariant components of (h), and we multiply (10) by $\lambda_{h|p}$ and sum for h, we have

$$R_{pr} = - \sum_h \rho_h\lambda_{h|p}\lambda_{h|r},$$

which is the form given by Ricci as characteristic of principal directions.

In order that the principal directions be completely indeterminate at every point, it is necessary that the coefficients of the λ's in (10) be zero; that is

$$R_{pr} = \varphi g_{pr}, \tag{12}$$

where φ is a scalar. When this is satisfied, we have from (9) that ρ_h is the same for all directions. Consequently the space may be thought of as homogeneous, and (12) is the necessary and sufficient condition for such homogeneity.

From (12) we have

$$R = g^{pr}R_{pr} = n\varphi,$$

and (12) becomes

$$R_{pr} = \frac{1}{n}\, g_{pr}R. \tag{13}$$

The original Einstein equations (1914) for space free from matter are those for which $\varphi = 0$ in (12); in 1917 (*Sitz. Pr. Ak. Wiss.*, Feb. 8), those for which $\varphi =$ const.; and in 1919 (*Ibid.*, Apr. 10), the general case (13) of a homogeneous space from the above point of view.

From (8), (9) and (12) it follows that for any three mutually orthogonal directions in any 3-space, which is homogeneous,

$$r_{12} + r_{13} = r_{21} + r_{23} = r_{31} + r_{32} = -\varphi,$$

whence

$$r_{12} = r_{13} = r_{23} = -\frac{\varphi}{2}.$$

Thus the Riemann curvature at each point is the same for all directions, and by the theorem of Schur (*Math. Ann.*, **27,** 563) is constant. Consequently the first type of Einstein space is a generalization of euclidean 3-space, and his other two spaces of 3-space of constant curvature.

NOTE ON THE DEFINITION OF A LINEAL FUNCTIONAL

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A linear functional is usually defined as one that is distributive and continuous, but the term continuous functional has been used in at least two ways which are not equivalent. F. Riesz, Fréchet, G. C. Evans and others have defined a continuous functional as one which satisfies the equation

$$\lim_{n \to \infty} L(u_n(x)) = L(u(x)), \tag{1}$$

when the sequence $u_1(x)$, $u_2(x)$, ... approaches $u(x)$ uniformly, while Levy[1] and W. L. Hart[2] have simply assumed that the sequence of u_n's converge in the mean. In Levy's paper the functional also depends on a parameter which has the same range as x, and $L(u_n(x),y)$ is only required to converge to $L(u(x),y)$ in the mean, but such a parameter will not be introduced here.

In what follows a distributive functional will be called linear when equation (1) is satisfied for uniformly convergent sequences of u's, and linear-*m* when they are only required to converge in the mean. If then L is to be linear it is necessary and sufficient that the equation

$$L(au_1 + bu_2) = aL(u_1) + bL(u_2) \tag{2}$$

shall be satisfied, and that there be an M independent of u which satisfies the inequality

$$|L(u)| \leqq M.\,max|u|, \tag{3}$$

while if it is to be linear-m this inequality must be replaced by

$$|L(u)| \leqq M\sqrt{\int_a^b u^2(x)dx.} \tag{4}$$

F. Riesz has proved that every linear functional can be put into the Stieltjes form

$$L(u) = \int_a^b u(x)d\alpha(x), \tag{5}$$

where $\alpha(x)$ is independent of u and has finite variation.[3] Fréchet has reduced this to the form

$$L(u) = \sum_{n=1}^{\infty} A_n u(x_n) + \int_a^b u(x)\beta(x)dx + \int_a^b u(x)d\lambda(x), \tag{6}$$

where the A_n's are constants, the x_n's are the points where $\alpha(x)$ is discontinuous, $\beta(x)$ is the derivative of $\alpha(x)$ where the derivative exists, and $\lambda(x)$ is continuous, and has its derivative equal to zero excepting on a set of measure zero.[4] The principal object of the present note is to prove that when L is linear-m equation (6) becomes simply

$$L(u) = \int_a^b u(x)\beta(x)dx. \tag{7}$$

The class $[u(x)]$ will include all bounded, real-valued functions defined on (a, b), which are integrable with respect to a function of finite variation by Young's method of monotone sequences, beginning with the set of continuous functions.[5]

In the proof of equation (5) it was only assumed that $L(u)$ is defined for continuous u's, and to make the theorem apply to the larger class, L must always be defined in such a way that whenever a monotone sequence $\{u_n(x)\}$ approaches a discontinuous $u(x)$, $L(u_n)$ will approach $L(u)$. When this is done equation (6) will also be satisfied for the larger domain $[u(x)]$, since the limiting processes extending the definition of the Stieltjes integral (5) to the domain $[u(x)]$ will make each term of equation (6) approach the proper value. On the other hand if such a monotone sequence approaches $u(x)$, the equation

$$\operatorname*{limit}_{n \to \infty} \int_a^b (u_n - u)^2 dx = 0$$

must be satisfied.[6] Then if L is linear-m inequality (4) implies that $L(u_n - u)$ will approach zero, and $L(u_n)$ must approach $L(u)$.

Since a uniformly convergent sequence must converge in the mean, a linear-m functional must also be linear.

If $L(u)$ is linear-m, and $u(x)$ is defined as identically zero excepting at x_1 where it is equal to unity, equation (6) becomes $L(u) = A_1$, and inequality (4) implies that $A_1 = 0$. Similarly $A_2 = A_3 = \ldots\ldots = 0$. In other words $\alpha(x)$ must be continuous.

Since $\alpha(x)$ also has finite variation, it can be put into the form $\alpha(x) = \alpha_1(x) - \alpha_2(x)$, where α_1 and α_2 are continuous, have finite variation, and are monotone increasing. The derivatives of α_1 and α_2 will be called β_1 and β_2. A set E of measure zero will contain all points where either β fails to exist. Since no derivative number of a monotone increasing function can be negative, the functions $\lambda_i(x)$, defined by the equations

$$\lambda_i(x) = \alpha_i(x) - \int_a^x \beta_i(x)dx, \qquad (i=1,\ 2) \qquad (8)$$

must be monotone increasing,[7] and their derivatives will vanish excepting on E. If a $k_1 > 0$ is now given, there must be a sequence of non-overlapping open intervals, $B_1 = \sum_{n=1}^{\infty} I_n$ containing each point of E as an interior point of some I_n, such that the measure of B_1 is less than k_1. Then the variation of λ_i on B_1 is equal to its variation on (a, b),[8] which is $\lambda_i(b) - \lambda_i(a)$. The lower semicontinuous functions $u_1(x)$, $u_2(x), \ldots\ldots$ will now be defined by the equations

$u_n(x) = 1$ on the interior of the intervals $I_1, I_2, \ldots\ldots I_n$,
$u_n(x) = 0$ for all other values of x,

and consequently

$$\int_a^b u_n(x)d\lambda_i(x) = \text{Variation of } \lambda_i \text{ on } \sum_{m=1}^{n} I_m, \ (i=1, 2;\ n=1, 2, \ldots).$$

The limit $v_1(x)$ of this monotone sequence of u's, which vanishes excepting on B_1, must satisfy the equations

$$\int_a^b v_i(x)d\lambda_i(x) = \lambda_i(b) - \lambda_i(a). \qquad (i=1,\ 2)$$

If a sequence $k_1 > k_2 > \ldots\ldots \longrightarrow 0$ is chosen, beginning with the above k_1, the sequences of intervals $B_2, B_3, \ldots\ldots$ can be chosen in such a way that each contains E, every point of each is a point of the preceding, and the measure of B_j is less that k_j. Then the sequence $v_2(x), v_3(x), \ldots\ldots$, determined as $v_1(x)$ was, will be monotone decreasing, and its limit $v(x)$ will vanish excepting on a set of measure zero, and satisfy the equations

$$\int_a^b v(x)d\lambda_i(x) = \lambda_i(b) - \lambda_i(a). \qquad (i=1,\ 2) \qquad (9)$$

Since $v(x)$ vanishes excepting on a set of measure zero, the equations

$$\int_a^b v(x)\beta_i(x)dx = 0, \qquad (i=1,\ 2)$$

must be satisfied, and inequality (4) implies that $L(v) = 0$. But since $L(u)$ must satisfy equation (6) which is now equivalent to

$$L(u) = \int_a^b u(x)(\beta_1(x) - \beta_2(x))dx + \int_a^b u(x)d(\lambda_1(x) - \lambda_2(x)), \qquad (10)$$

it is necessary that

$$\int_a^b v(x)d(\lambda_1(x) - \lambda_2(x)) = 0.$$

Then equation (9) implies that $\lambda_1(b) - \lambda_2(b) = \lambda_1(a) - \lambda_2(a)$.

If in the above proof the functions $u_n(x)$, $v_j(x)$, and $v(x)$ had been defined as they were on the interval (a, y), where y is an arbitrary point on (a, b), and as identically zero on the rest of (a, b), the same argument would prove that $\lambda_1(y) - \lambda_2(y) = \lambda_1(a) - \lambda_2(a)$. That is, $\lambda_1 - \lambda_2$ is a constant, and equation (10) is equivalent to equation (7), with $\beta = \beta_1 - \beta_2$. This completes the proof of the theorem.

If L also depends on a parameter, y, and if instead of assuming that equation (1) is satisfied for every value of y, it is assumed that $L(u_n(x),y)$ approaches $L(u(x),y)$ in the mean, the theorem does not apply. For instance the functional

$$L(u(x),y) = u(y)$$

satisfies this condition, but cannot be represented by an integral analogous to (7).

[1] Levy, *Bul. Soc. Math. France*, **48,** 1920 (20).

[2] Hart, *New York, Bul. Amer. Math. Soc.*, **27,** 1921 (308).

[3] F. Riesz, *Ann. Sci. Ec. Norm., Paris* (Ser. 3), **31,** 1914 (9–14).

[4] Fréchet, *New York, Trans. Amer. Math. Soc.*, **15,** 1914 (152), or Evans, *Cambridge (Mass.), Colloquium Lectures* (60).

[5] Young, *Proc. London Math. Soc.*, **13,** 1914 (137), or Daniell, *Princeton, Ann. Math.*, **19,** 1917 (288–93).

[6] Bliss, *New York, Bul. Amer. Math. Soc.*, **24,** 1917 (24, corollary).

[7] *Ibid.* (38, Lemma 8).

[8] *Ibid.* (36–40, Theorems, 18 and 19).

SKIN TRANSPLANTATION IN FROG TADPOLES

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Communicated by G. H. Parker, December 21, 1921

1. The experimental inhibition of vision by means of opaque grafts over the eyes of frog tadpoles (from 20 to 100 mm. in length) should contribute evidence as to any regulatory interaction between the graft and the eye, tending to restore the function of the eye. For this purpose 202 operations were performed upon two species of Rana, catesbeiana and clamatans, which had been collected from three widely separated sources.

The first grafts consisted of tail skin placed over the eye. Sixty-six per cent of these showed an adjustment whereby a small part of the graft was absorbed in a manner which tended to restore vision. When back skin was used, no such absorption occurred. Skin from the tail or back, grafted on other parts of the body, persisted indefinitely. However, when tail skin was grafted over hemispheres of glass or celloidin with radii equal to the radius of the eyeball, absorption occurred. The process in these grafts over "artificial eyes" was like that over normal functional eyes. Back-skin grafts so placed were not absorbed. As a check experiment, to determine whether or not the presence of the foreign material caused the absorption, other operations were made in which thin plates of glass cut from the ordinary cover-slip were used instead of the hemispheres. Such grafts were not absorbed, proving that the foreign material was not the cause of absorption.

From a study of the transplants, it was concluded that the curvature of the eyeball and of the artificial eye is the cause of absorption of tail-skin grafts. The plastic tail skin yields to the tension produced by the curvature and adjusts itself. In the majority of cases, this adjustment by absorption of tissue incidentally restores vision. It always ceases as soon as the tension is relieved. When the graft is very thick or is unusually well attached to the host, no adjustment is possible. Nor is it possible in back-skin grafts, because of their more compact structure.

The proliferation of new tissue by the transplants was limited to the tail-skin grafts. In some cases miniature tail tips were formed, while in others irregular proliferation predominated. This latter type is called "amorphic regeneration." The original anterior end, as well as the posterior end, of the graft is able to produce a new notochord and nerve cord in the regenerated tissue. It was found that the absence of light hastens the proliferation period by about nine days, and slightly increases the rate of proliferation. "Dark" grafts regularly showed amorphic regeneration. Following the period of proliferation, all grafts entered a state of equilibrium. In none of the grafts taken from the back or the belly was proliferation observed.

The conclusion is reached that no functional regulation of skin grafts over the eyes occurs. The adjustment which does take place is purely mechanical and its relation to the visual function is quite incidental. The lack of correlative regulation is probably the result of the high state of differentiation and specificity attained by the eye and the skin.

2. When integument from the tail, belly, or back is transplanted to another region on the same animal (auto-transplant), it preserves its characteristic appearance and produces new tissue of its own kind indefinitely. An homoio-transplant (from another individual not closely related) preserves its integrity only temporarily, being ultimately replaced

by regenerated host tissue. From these facts it is concluded that the integument of frog tadpoles is locally specific, and self-differentiating even when transferred to a new soil on the same animal.

3. Auto- and homoio-transplants of white belly skin on the dark back regions always acquire melanophores. The process varies according to the source of the transplant. In auto-transplants there is no antagonistic reaction between the protoplasms of the graft and the host, and union is quickly established. Melanophores appear only after several weeks, sometimes months. They are accompanied by the appearance of an abundance of scattered melanin granules and irregular masses of pigment, which may readily be interpreted as due to the presence of "young" melanophores. In homoio-transplants, union is delayed by the action of the homoio-toxin. In an effort to cover the wound as soon as possible, the surrounding epidermal cells migrate rapidly over the wound, and over the graft as though it were a foreign body. Melanophores are carried along. Such epidermal pigmentation may occur within twenty-four hours. Subsequent increase in the number of epidermal melanophores and the appearance of dermal melanophores and xantholeucophores are caused by formation *in situ* as a result of the normal growth activity of the integument.

It may therefore be said that in auto-transplants the acquisition of melanophores is chiefly the result of formation of pigment from the epithelial cells *in situ*, while in homoio-transplants the pigmentation is chiefly the result of epidermal migration which carries along melanophores. It may be followed by formation *in situ*.

4. When the conjunctiva of frog tadpoles is extensively injured by scratching or pricking, the regenerated tissue is pigmented first by epidermal melanophores and later by dermal melanophores and xantholeucophores. Such pigmentation, which may obscure the eye and presumably impair vision, persists indefinitely. It is further proof that no regulatory action between the eye and the overlying skin exists in tadpoles over 20 millimeters in length. The method of appearance of the pigment throws further light on the question of epidermal wound healing. The extensive injury of the conjunctiva compels the surrounding skin to repair the wound. The epidermal cells migrate over it and carry along the melanophores. Subsequent increase of pigment results from mitotic multiplication of integumentary cells.

5. A series of experiments was performed to discover the stimuli causing expansion and contraction of the dermal melanophores. Briefly, it was found that expansion is caused by a low temperature (near 0° C.) independently of illumination; by darkness; by a 0.1% chloretone solution; by anoxemia and by a low metabolic rate coincident with a moribund state. Contraction of the dermal melanophores is caused by a high temperature (near 35° C.) independently of illumination; by light, and

by a return of normal environmental conditions after removal of the effects of chloretone and anoxemia. Recovery from a moribund state also is accompanied by contraction.

Either expansion or contraction usually requires about thirty-six hours for completion, although the reactions begin within twenty minutes after initiation of the stimulus. Oxygen deficiency brings about complete expansion in about one-half hour. It is probable that any stimulus which lowers the metabolic rate of the whole animal also causes expansion of the melanophores, and any stimulus causing an increase in the metabolic rate results in contraction of the melanophores.

The complete account of these investigations will appear shortly in the *Journal of Experimental Zoölogy.*

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ON THE RELATION OF A CONTINUOUS CURVE TO ITS COMPLEMENTARY DOMAINS IN SPACE OF THREE DIMENSIONS[1]

By Robert L. Moore

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Communicated by E. H. Moore, February 1, 1922

Schoenflies has shown[2] that, in order that a closed, bounded and connected point-set, lying in a *plane* S, should be a continuous curve, it is necessary and sufficient that (1) if R is a domain complementary to M, every point of the boundary of R should be "accessible from all sides" with respect to R, and (2) if ϵ is any preassigned positive number, there do not exist infinitely many distinct domains, complementary[3] to M, and all of diameter[4] greater than ϵ. In the present paper I will exhibit examples of continuous curves, in *space of three dimensions*, which satisfy neither of these conditions.

If, in a *plane* S, a closed and bounded point-set K separates S into just two domains, S_1 and S_2, such that every point of K is a limit point both of S_1 and of S_2, then, in order that K should be a simple closed curve, it is[5] necessary and sufficient that every point of K should be accessible[6] from every point which does not belong to K. In this paper it will be shown that, for a space S of *three dimensions*, this condition is neither necessary nor sufficient in order that K should be a simple closed surface or in order that it should be a continuous curve. It will be shown, however, that a closed, bounded and connected point-set in space of three dimensions is a continuous curve provided it is the common boundary of two mutually exclusive domains both of which are *uniformly connected im kleinen.*[7]

1. *There exists, in a three dimensional space S, a continuous curve M which divides S into just two domains, S_1 and S_2, such that* (*a*) *the boundary of S_1 contains a point which is not accessible from any point of S_1,* (*b*) *the domain S_2 is uniformly connected im kleinen,* (*c*) *the boundary of S_1 is not a continuous curve.*

Example.—Let K denote a cube placed so that b, one of its bases, is horizontal. Let l and f denote two opposite lateral faces of K. Let

$ABCD$ and $EFGH$ denote two squares concentric with l and lying within it and such that $EFGH$ is within $ABCD$, AB and EF being horizontal. Let I and J denote points in the face f such that the planes IAB and JCD are parallel to b and such that I is equidistant from A and B, while J is equidistant from C and D. We thus have a right prism whose triangular bases are ABI and DCJ. Let P denote this prism and let Q denote the right pyramid whose base is the square $EFGH$ and whose apex is at O, a point half-way between I and J. For each positive integer n, let α_n denote a plane parallel to f and at a distance from f equal to a/n, where a is one-half the length of a side of K. For each n, let β_n denote a plane parallel to, and half-way between the planes α_n and α_{n+1}. Let I_n denote that portion of the interior of P which is intercepted between the planes α_n and β_n. Let R denote the interior of the pyramid Q. Let S_1 denote the point-set composed of R together with the sets $I_1, I_2, I_3, I_4, \ldots$. Let N denote the point-set composed of the surface of K together with its interior and let M denote N-R. The set M is a continuous curve, S_1 is one of its complementary domains, and the straight line interval JI is a portion of the boundary of S_1. But no point of JI, except O, is accessible from any point of S_1. The boundary of S_1 is not connected *im kleinen* at any point of JI except O. Clearly S_2, the exterior of the cube K, is uniformly connected *im kleinen*.

2. *If ϵ is any preassigned positive number, there exists a continuous curve M such that there are infinitely many distinct domains, complementary to M and all of diameter greater than ϵ.*

Example.—Consider again the cube K and the domains $I_1, I_2, I_3, \ldots$ of Example 1, it being understood that the distance between the planes IAB and JCD of that example is 2ϵ. Let M denote the point-set $N-(I_1+I_2+I_3+\ldots)$ where N has the same meaning as in Example 1. Here M is a continuous curve and each I_n is one of its complementary domains.

3. *There exists, in three dimensional space, a continuous curve M which divides space into just two domains, S_1 and S_2, such that (a) every point of M is a limit point both of S_1 and of S_2, (b) not every point of M is accessible from a point of S_2.*

Example.—Let K denote a square whose vertices A, B, C, D are the points $(0, -1, -1)$, $(0, 1, -1)$, $(0, 1, 1)$ and $(0, -1, 1)$, respectively. For each integer n (positive or negative) let K_n denote a square whose vertices, A_n, B_n, C_n, D_n, are the points $(1/n, -1, -1)$, $(1/n, 1, -1)$, $(1/n, 1, 1)$ and $(1/n, -1, 1)$, respectively. For each n let $\overline{K}_n$ denote a square whose vertices, $\overline{A}_n, \overline{B}_n, \overline{C}_n, \overline{D}_n$, are so situated that $\overline{A}_n$ is half-way between A_n and A_{n_1}, $\overline{B}_n$ is half-way between B_n and B_{n_1}, $\overline{C}_n$ is half-way between C_n and C_{n_1} and $\overline{D}_n$ is half-way between D_n and D_{n_1}, where n_1 denotes $n+1$ or $n-1$ according as n is positive or negative. For every integer n and every pair of integers i and j which are numerically less than $4^{|n|}$, let $\overline{P}_{ijn}$ denote that point, within the square $\overline{K}_n$, whose y and z are $i/4^{|n|}$ and $j/4^{|n|}$, respectively. For each n, let T_n denote the set of all points $[\overline{P}_{ijn}]$ for which i and j are even numbers numerically less than $4^{|n|}$ and let U_n denote the set of all points $[\overline{P}_{ijn}]$ such that i and j are odd numbers numerically less than $4^{|n|}$. For every point $\overline{P}_{ijn}$ belonging to T_n or to U_n let $\overline{K}_{ijn}$ denote a circle lying in the plane $\overline{K}_n$, with center at $\overline{P}_{ijn}$ and with a radius equal to $1/4^{|n|+2}$. Thus with each point of the set $T_n + U_n$ there is associated a definite circle having that point as center, and every one of these circles lies wholly without every other one. Now let P'_{ijn} and K'_{ijn} denote the orthogonal projections of P_{ijn} and $\overline{K}_{ijn}$, respectively, onto the plane of the square K_{n_1} and let P_{ijn} denote the orthogonal projection of $\overline{P}_{ijn}$ onto the plane of K_n. For each point $\overline{P}_{ijn}$ belonging to the set T_n let S_{ijn} denote the right circular cylinder whose bases have the circles $\overline{K}_{ijn}$ and K'_{ijn} as their perimeters, and for each point $\overline{P}_{ijn}$ belonging to U_n let C_{ijn} denote the right circular cone whose base has $\overline{K}_{ijn}$ as its perimeter and whose apex is the point P_{ijn}. It is to be noted that S_{ijn} exists only when i and j are even and that C_{ijn} exists only when i and j are odd. For each n, let P_n denote the right parallelopiped which has K_n and $\overline{K}_n$ as opposite lateral faces and let R_n denote the interior of P_n. Let I_n denote that portion of R_n which remains after the removal of every point of R_n which lies either within, or on the surface of any C_{ijn} (i and j odd). For each n let L_n denote the point-set obtained by adding to I_n the interiors of all the cylinders S_{ijn} (i and j even) together with the interiors of the bases of these cylinders. Let K^* denote a rectangle lying in the plane $z = 2$ and with sides equal and parallel to the respective sides of the upper base of P_1. Let P' denote a parallelopiped whose upper base is K^* and whose lower base is the upper base of P_1. Let P'' denote a parallelopiped whose upper base is K^* and whose lower base is the upper base of P_{-1}. Let L^* denote the point-set composed of the interiors of P' and P'' and the interiors of the upper bases of P_1 and P_{-1}. Let H denote the point-set $L^* + L_1 + L_2 + L_3 + \ldots$, and let M denote its boundary. The point-set H is a domain and M is a continuous curve which separates space into two domains S_1 and S_2 (where S_1 is H). Every point within the rectangle K belongs to M. But no point within this rectangle is accessible from any point in S_2, though every point of M is a limit point both of S_1 and of S_2. Neither S_1 nor S_2 is uniformly connected *im kleinen*.

4. *There exists, in a space S of three dimensions, a closed, connected and bounded point-set M which divides S into just two domains such that* (*a*) *both of these domains are uniformly connected im kleinen, and every point of M is accessible from one of these domains, but* (*b*) *M is not a continuous curve.*

Example.—Let K denote the cube which is bounded by the planes $x = 0$,

$x=2, y=0, y=2, z=0, z=2$. For each positive integer n let A_n denote the point $(1/n, 1/n, 0)$ and let B_n denote the point $(1/n, 1/n, 2)$. Let M denote the point-set composed of the surface of K together with all the straight line intervals $A_1 B_1, A_2 B_2, A_3 B_3, \ldots$.

5. *There exists, in three dimensional space, a closed, bounded and connected point-set M which is not a continuous curve but which divides space into just two domains* S_1 *and* S_2 *such that* (*a*) *every point of M is accessible both from every point of* S_1 *and from every point of* S_2, (*b*) *the domain* S_1 is uniformly connected *im kleinen.*

Example.—Returning to Example 3, let H denote a cube whose upper base lies in the plane $z=1$, whose center is at the point $(0, 0, -1)$, and one of whose faces is in the plane $x=2$. For every positive integer n, the upper base of the parallelopiped P_n of Example 3 is a part of the upper base of H but the remainder of P_n is wholly within H. Let R denote the set of points composed of (*a*) the exterior of H, (*b*) the interiors of the upper bases of the parallelopiped $P_1, P_2, P_3, \ldots$, (*c*) the interiors of $P_1, P_2, P_3, \ldots$ and (*d*) the interiors of those cylinders S_{ijn} (of Example 3) for which n is positive, together with the interiors of the bases of those cylinders. Let M denote the boundary of R. The point-set M is closed, connected and bounded and it divides space into just two domains, S_1 and S_2 (where S_1 is R), and furthermore every point of M is accessible from every point of S_1 and from every point of S_2. But M is not a continuous curve. It fails to be connected *im kleinen* at the point $(0, 0, 0)$. The domain S_1 is uniformly connected *im kleinen* but the domain S_2 is not.

THEOREM 1. *If, in a three dimensional space S, the closed, bounded and connected point-set M divides S into just two domains* S_1 *and* S_2 *such that* (*a*) *every point of M is a limit point both of* S_1 *and of* S_2, *and* (*b*) *both* S_1 *and* S_2 *are uniformly connected im kleinen; then M is a continuous curve.*

This theorem will be proved with the assistance of the following lemma. A proof of this lemma will be given elsewhere.

Lemma. *Suppose that, in space of three dimensions,* (*1*) *M is a closed point-set,* (*2*) $A_1 B_1$ *and* $A_2 B_2$ *are two simple continuous arcs each of which consists of a finite number of straight line intervals,* (*3*) *neither of the arcs* $A_1 B_1$ *and* $A_2 B_2$ *has a point in common with M,* (*4*) *every simple continuous arc which joins a point of* $A_1 B_1$ *to a point of* $A_2 B_2$ *contains at least one point of M. Then M contains a closed and connected point-set which contains at least one point in common with each of the straight line intervals* $A_1 A_2$ *and* $B_1 B_2$ *and has a diameter less than, or equal to, twice that of the point-set composed of the two arcs* $A_1 B_1$ *and* $B_1 B_2$.

Proof of Theorem 1.—Suppose, on the contrary, that M is not connected *im kleinen.* Then there exists a positive number ϵ and two infinite sequences of points $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$, belonging to M, such that $\lim_{n=\infty} A_n B_n = O$ and such that, for no n, do A_n and B_n lie together in a closed and connected subset of M of diameter less than or equal to ϵ. But, since the domains S_1 and S_2 are uniformly connected *im kleinen*, there exists a positive number $\delta\epsilon$ such that if X and Y are two points both belonging to S_1, or both belonging to S_2, and such that $XY < \delta\epsilon$ then X and Y can be joined by a simple broken line of diameter less than $\epsilon/9$ which lies wholly in S_1 or wholly in S_2. There exists an integer m such that A_m, $B_m < \delta\epsilon/3$. For each positive integer n there exists spheres α_n and β_n, with centers at A_m and B_m, respectively, and with diameters less than each of the numbers $\delta\epsilon/3$, $\epsilon/9$ and $1/n$. There exist, within α_n, points E_n and E'_n belonging to S_1 and S_2, respectively, and within β_n, points D_n and D'_n belonging to S_1 and S_2, respectively. There exist broken lines $D_n\ E_n$ and $D'_n\ E'_n$ lying in S_1 and S_2, respectively, and with diameters less than $\epsilon/9$. Every simple continuous arc from a point of $D_n\ E_n$ to a point of $D'_n\ E'_n$ contains a point of M. It follows from the above lemma that M contains a closed and connected point-set M_n, of diameter less than or equal to $2\epsilon/3$, which contains a point within α_n and a point within β_n. The sequence of closed and connected point-sets M_1, M_2, M_3, ... has, as its limiting set,[8] a closed and connected subset of M which contains both A_m and B_m and has a diameter less than ϵ. Thus the supposition that M is not connected *im kleinen* has led to a contradiction.

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[1] Presented, in part, to The American Mathematical Society, October 29, 1921.

[2] Schoenflies, A., *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Zweiter Teil, Leipsig, 1908, p. 237.

[3] A connected set of points M lying in a space S of one or more dimensions is said to be a *domain* (with respect to S) provided there exists, in S, a closed (or vacuous) point-set N such that $M=S-N$. The domain R is said to be complementary to the closed point-set M if (1) R contains no point of M, and (2) the boundary of R is a subset of M.

[4] The diameter of a bounded point-set M is the smallest number d such that if X and Y are any two points of M then the distance from X to Y is less than d.

[5] Cf. Schoenflies, A., "Ueber einen grundlegenden Satz der Analysis Situs," *Göttingen, Nachr. Ges. Wiss.*, 1902 (185).

[6] A point X, belonging to a closed point-set K, is said to be *accessible* from a point Y which does not belong to K if there exists a simple continuous arc from X to Y which has no point except X in common with K.

[7] A point-set M is said to be *connected "im kleinen"* if, for every point P of M and every positive number e, there exists a positive number δ_{ep} such that if X is a point of M at a distance less than δ_{ep} from the point P, then X and P lie together in a connected subset of M every point of which is at a distance of less than e from the point P. The set M is said to be *uniformly* connected *im kleinen* if for every positive number e there exists a positive number δ_e such that if P_1 and P_2 are two points of M at a distance apart less than δ_e then they lie together in a connected subset of M every point of which is at a distance of less than e from P_1. Cf. Hahn, H., *Wien. Ber.*, **123**, 1914 (2433). Hahn shows that in order that a closed, bounded and connected point-set M should be a continuous curve it is necessary and sufficient that M should be connected *im kleinen*. See also, however, Mazurkiewiez, S., *Fundamenta Mathematicae*, **1**, 1920 (166–209). In this article the author refers to earlier articles by himself, published, in 1913 and 1916, in a journal (*C. R. Soc. Sc. Varsovie*) to which I do not, at present, have access.

[8] The point-set t is said to be the limiting set of the sequence of point-sets t_1, t_2, t_3, . . . provided that (a) each point of t is the sequential limit point of an infinite subsequence of some sequence of points, P_1, P_2, P_3, . . . sucht hat, for every n, P_n belongs to t_n, and (b) if P_1, P_2, P_3, . . . is a sequence of points such that, for every n, P_n belongs to t_n then t contains the sequential limit point of every subsequence of P_1, P_2, P_3, . . . which has a sequential limit point.

CHARACTERISTIC EFFECTS UPON GROWTH, OESTRUS AND OVULATION INDUCED BY THE INTRAPERITONEAL ADMINISTRATION OF FRESH ANTERIOR HYPOPHYSEAL SUBSTANCE*

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Communicated by L. Hektoen, December 23, 1921

At the last session (Proc. Amer. Assoc. Anat., *Anat. Rec.*, vol. 21) we reported a characteristic acceleration of growth in rats treated intraperitoneally with the finely ground, fresh anterior lobe of the hypophysis of beef. We had shown that this effect is in marked contrast to the lack of effects from oral administration of the same substance, even in very large amounts. Two careful series of experiments with litter mate controls have now been completed, some members of the first series having been experimented with continuously for a period of over one year. *Such animals are invariably much heavier than their litter mate sisters.* The greatest disparity which has been observed was attained on the 333rd day of life when an animal receiving anterior hypophyseal substance weighed 596 grams and its healthy litter mate control weighed 248 grams. It would not appear to be incorrect to characterize these changes as producing constantly a certain degree of true gigantism. Increase in weight results to a great extent from a storage of fat, but is not solely due to this, the skeleton being invariably somewhat larger and heavier, and, as would be expected, the heart, lung, alimentary canal and kidney are heavier. The fat deposits in the omentum and mesentery are such that these structures weigh three or four times as much in the experimental animals as in their controls. The weight of the hypophysis, thyroid and thymus is not appreciably effected.

Most surprising is the effect on the reproductive system. *Oestrus, as detected by typical changes in the vaginal smear, may never occur in these animals or may be exhibited only at long intervals.* It was, consequently, surprising to find that in all instances the ovaries instead of being underdeveloped weighed twice as much as they did in the control animals and exhibited great numbers of substantial corpora lutea. The uterus, on the contrary, weighed absolutely about half as much as it did in the normal controls. Histological examination of the gonads confirmed the presence of very abundant lutein tissue and demonstrated the formation of this tissue about the egg in unruptured, normal follicles and in atretic follicles. Ripe, normal Graafial follicles were invariably absent. *A powerful, specific stimulus to lutein cell transformation has thus been effected by this hormone.*

Aside from the fact that it is tolerated only in smaller quantities, fresh posterior hypophyseal substance gave none of the above characteristic effects upon growth, oestrus and ovulation in animals as compared with similar care with litter mate sisters.

* Aided by grants from the Committee on Scientific Research of the American Medical Association.

A STUDY OF THE EFFECTS OF CUCURBITA PEPO SEEDS ON KIDNEY EXCRETION

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Communicated by R. Pearl, February 4, 1922

1. *Purpose and Plan of Experiment.*—The object of this experiment was to compare and verify the results obtained from a somewhat similar experiment conducted during March 1920.[1] Moreover, a further study of the effects of Cucurbita seeds on kidney excretion is thereby promulgated.

Since the previous experiment does not give sufficient analytical data so far as the urine samples are concerned a more elaborate plan was followed by virtue of which not only comparative data may be obtained but additional analyses may be available for the elucidation of the problem under consideration.

Three successive dietary periods were arranged for the experiment, namely, three days of a preliminary basal dietary period, three days of an experimental dietary period, and three days of a final dietary period. The entire experiment lasted for nine consecutive days.

All the foodstuffs consumed in this experiment were measured. The pumpkin seeds (Cucurbita pepo) were of two kinds, roasted and raw, a known quantity of each kind was added to each meal during the experimental dietary period.

2. *Dietary.*—Hindhede's "back-to-the-farm" dietary,[2] slightly modified, was used for the basal diet. Composition of this dietary is given in table 1. This diet was used during the preliminary and final periods of this experiment. The experimental diet consisted of the same basal diet plus 30 grams of roasted pumpkin seeds for the first day; 35 grams of roasted and 30 grams of raw pumpkin seeds mixed for the second day, and 30 grams of raw seeds for the third day of the experimental dietary period.

TABLE 1

Composition of the Basal Dietary[3]

NAME	QUANTITY	PROTEIN	FAT	CARBOHYDRATES	CALORIES
Water	600 cc.				
Graham bread	300 gms.	19.7	4.8	153.9	780
Apples	600 gms.	1.8	3.0	76.8	342
Milk	800 cc.	25.6	30.4	40.0	620
Butter	60 gms.	.6	48.4	...	450
Potatoes	800 gms.	15.2	.8	160.0	800
Total		62.9	87.4	430.7	2992

No salt was used.

3. *Period I.*—In securing the daily samples of urine the bladder was emptied about 7 a. m., April 3rd, and this urine was discarded. All the urine from that hour up to and including that passed the next day at 7 a. m. was saved, thoroughly mixed, preserved with 5 cc. of toluene, a sufficient amount to film the bottom and the walls of the container. The latter were wide mouth glass jars with ground glass stoppers. Thus, each composite sample represented the mixed excretions for twenty-four hours.

As soon as a twenty-four hour sample was secured it was immediately analyzed for the determination of volume, specific gravity, temperature, titratable acidity (using 0.1 *N* NaOH) (4, pp. 508–509), NH_3 determination (Folin's method) (4, p. 523); total nitrogen determination (Kjeldahl method) (4, pp. 512–513); creatinine (Folin's colorimetric method) (4, pp. 530–531); urea (Marshall's method) (4, p. 522); uric acid (Folin and Schaffer method) (4, p. 536); and total solids of a sample were computed by means of Long's coefficient (4, p. 512). The remainder of each of the samples was then covered with a layer of toluene to preserve it for the inorganic analyses.

TABLE 2*a*

Analyses of Composite Samples of Urine from Period I

DATE 1921	VOLUME CC.	TEMP. C.	SP. GR.	T. S. GMS.	ACIDITY CC. 0.1 *N* NaOH PER 100 CC.	TOTAL N GMS.
Apr. 3	1360	24	1.019	65.77	39.0	11.72
Apr. 4	1600	22	1.017	66.56	24.3	9.85
Apr. 5	1710	24	1.016	69.35	10.8	10.04

TABLE 2*b*

Nitrogenous Ingredients Found in Samples of Urine, Period I

DATE 1921	AMMONIA	CREATININE	URIC ACID	UREA	UNDETERM. N
Apr. 3	.38	1.07	.57	20.28	1.37
Apr. 4	.48	1.24	.65	17.17	.78
Apr. 5	.31	1.28	.73	18.54	.39

4. *Period II.*—The pumpkin seeds used during this experiment were obtained from a seed grower. The roasting was done after the hulls were removed. The hull-free seeds were placed in a flat glass dish and kept in a dry-heat oven for three hours at a temperature of 110° F. or 43° C.

The table below shows the quantities of pumpkin seeds ingested at each meal during the experimental Period II.

	APRIL 6			APRIL 7			APRIL 8		
KIND	BREAKF. GMS.	LUNCH GMS.	DIN. GMS.	BREAKF. GMS.	LUNCH GMS.	DIN. GMS.	BREAKF. GMS.	LUNCH GMS.	DIN. GMS.
Roasted	5	10	15	15	15	5	..	..	.
Raw p. s.	.	..	..	5	10	15	15	10	5

The same methods of sampling and analyzing employed in Period I were also used in Period II of this experiment.

TABLE 3*a*

ANALYSES OF COMPOSITE SAMPLES OF URINE FROM PERIOD II

DATE 1921	VOLUME CC.	TEMP. C.	SP. GR.	T. S. GMS.	ACIDITY CC. 0.1 *N* NaOH PER 100 CC.	TOTAL N GMS.
Apr. 6	1425	20	1.019	64.46	17.8	9.69
Apr. 7	1945	19	1.013	55.62	15.1	10.14
Apr. 8	2150	22	1.014	72.67	14.2	12.08

TABLE 3*b*

NITROGENOUS INGREDIENTS FOUND IN SAMPLES OF URINE FROM PERIOD II

DATE 1921	AMMONIA	CREATININE	URIC ACID	UREA	UNDETERM. N
Apr. 6	.41	.79	.74	17.69	.58
Apr. 7	.44	1.01	.79	17.88	.76
Apr. 8	.56	1.34	.85	20.61	1.19

5. *Period III.*—During the final three days, April 9, 10, 11, of this experiment the diet was exactly the same as during the preliminary Period I.

TABLE 4*a*

ANALYSES OF COMPOSITE SAMPLES OF URINE FROM PERIOD III

DATE 1921	VOLUME CC.	TEMP °C.	SP GR	T. S. GMS.	ACIDITY CC. 0.1 *N* NaOH PER 100 CC.	TOTAL N GMS.
Apr. 9	1705	22	1.016	66.49	20.5	10.16
Apr. 10	2370	17	1.015	76.40	7.5	9.70
Apr. 11	1910	18	1.016	67.53	12.8	9.45

TABLE 4*b*

NITROGENOUS INGREDIENTS FOUND IN SAMPLES OF URINE, PERIOD III

DATE 1921	AMMONIA	CREATININE	URIC ACID	UREA	UNDETERM. N
Apr. 9	.53	1.18	.62	16.58	1.37
Apr. 10	.55	1.35	.63	17.26	.49
Apr. 11	.50	1.32	.66	15.63	1.04

6. *Discussion.*—To reduce the effect of the variation factor upon the study of results and to obtain a more satisfactory comparison the hypothetical mean of results of each period was computed.

The following table V presents the data for comparison.

TABLE 5

COMPARATIVE RESULTS FOUND DURING PERIODS I, II AND III
(HYPOTHETICAL MEANS)

PERIOD	VOLUME CC.	T. S. GMS.	ACIDITY CC. 0.1 *N* NaOH	T. N. GMS.	NH_3 GMS.	CREATININE GMS.	URIC ACID, GMS.	UREA GMS.
I	1655	67.95	17.5	9.94	.39	1.26	.69	17.85
II	2047	64.14	14.6	11.11	.50	1.17	.82	19.24
III	2140	71.96	10.15	9.57	.52	1.33	.64	16.44

From this table we find that the average daily output of urine during Period II was greater than that of Period I but smaller when compared with the average daily volume of urine excreted during Period III.*

The amount of total solids of the experimental Period II was less as compared with either of the basal periods (I and II) which served as a check in this work. The acidity of the urine excreted during Period II was on the average slightly greater than that of Period III which agrees with the results obtained in the first experiment.[1]

The total nitrogen output is highest in Period II due to the fact that additional feeding substances, in the form of pumpkin seeds, were added to the basal diet.

Of the nitrogenous ingredients of the average samples of the three periods, uric acid of Period II shows a marked relation to that of Periods I and III. Namely, taking the average quantity of uric acid excreted during Period III as 100%, we find that the average output of uric acid during Period II was 128.12%.

Creatinine of the average sample of urine excreted during Period II as compared to that of Period III is as 100% is to 87.97%. The excretion of urea during Period II was 117%.

A comparative qualitative test (Obermayer's) for indican of all the nine samples of urine gave the results shown in table 6 below:

TABLE 6

COMPARATIVE INDICAN CONTENT

SAMPLE NO.	GRAPHIC REPRESENTATION	RANK ACCORDING TO INTENSITY OF COLOR
1	******	4th place
2	***	7th place
3	**	8th place
4	****	6th place
5	*******	3rd place
6	*********	1st place
7	********	2nd place
8	*****	5th place
9	*	9th place

It is of significant interest to note that the last two samples of the experimental period and the first sample of the final period gave the deepest coloration when tested for indican.

7. *Inorganic Determinations.*—For the determination of total sulphates and ethereal sulphates Folin's methods were employed; total phosphates were determined by the Uranium Acetate method; for the determination of chlorides and chlorin Volhard-Arnold method was employed.

TABLE 7

MINERAL INGREDIENTS FOUND IN THE URINE SAMPLES, GMS.

SAMPLE NO.	CHLORIDES	Cl	TOTAL P_2O_5	TOTAL SO_3	ETHEREAL SO_3
1	9.05	5.43	1.67	1.57	.268
2	8.66	5.20	1.56	1.60	.367
3	7.42	4.45	1.69	1.64	.328
4.6	6.61	3.96	1.71	1.76	.332
5	6.51	3.90	1.87	1.78	.346
6	7.86	4.72	2.08	1.97	.426
7	6.83	4.10	1.73	1.70	.331
8	7.77	4.66	3.14	1.83	.299
9	7.28	4.39	1.74	1.83	.288

8. *Conclusions.*—I. The results obtained in the experiment substantiate the results of the first experiment inasmuch as the volume and total solids output during the experimental Period II was less than in the check Period III; the hypothetical volume for the check period being 2140 cc. while during the experimental Period II the output was 2047 cc.; the average output of total solids during the check period being 71.96 while during the experimental Period II it was 64.14 gms.

II. There was 28.12% more uric acid eliminated during Period II than during the check Period III.

III. Also 17% more urea was eliminated during the experimental period than during the check Period III.

IV. The striking difference in the intensity of color in the urine samples of the experimental period revealed by the indican test is of pathological significance.

V. From the verified results and additional data obtained in this experiment the rôle played by Cucurbita pepo seeds in animal metabolism is of a chemico-pharmacognostic value.

I wish to express my thanks to Prof. W. J. Gies (Columbia) and Dr. E. G. Miller, Jr. (Columbia), for helpful suggestions in this work and to Dr. J. G. Lipman, Director of N. J. Agr. Exp. Station, for material assistance with this experiment.

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* For the sake of uniformity in this work the mean of only the second and third days' results of each period was taken. For it is evident that the first day's excretions are only slightly affected by the change of diet.

[1] Masurovsky, B., "Effects of Some Cucurbita Seeds on Animal Metabolism," *J. Agric. Res.*, **21**, No. 8, 1921 (523).

[2] Lusk, G., *Science of Nutrition*, p. 340.

[3] Atwater and Bryant, "The Chemical Composition of American Food Materials," *Bull.* **28** (Revised), *U. S. Dept. of Agriculture*, 1902.

[4] Hawk, P. B., *Practical Physiological Chemistry*, 6th edition, 7 pl. (col.), 1920.

THE ACTION OF DIAZOMETHANE ON URACIL

By Treat B. Johnson, Arthur J. Hill and Francis H. Case

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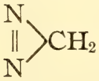
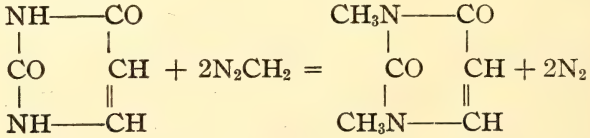
Communicated Jan. 25, 1922

In the course of work now in progress on the determination of constitution of nucleic acid and its primary hydrolytic products—neucleotides and neucleosides—it has finally become necessary to develop and utilize a new method for introducing methyl radicals into pyrimidine and purine combinations. The alkylation methods generally employed for bringing about such changes, namely, by interaction of alkali or silver salts with methyl iodide and dimethyl sulphate, have not proven satisfactory, and it has become necessary to turn to a reagent that will interact directly with the pyrimidine nucleus at a low temperature and in other solvents than water or alcohol. A reagent which has found little use thus far in biochemical research and which gives promise of proving extremely helpful in the solution of our problem is the very reactive aliphatic reagent, diazomethane:



We now find that pyrimidines of the type of uracil interact smoothly with this reagent in ether solution at ordinary temperature. The reaction is productive of nitrogen-alkyl derivatives and in the case of the pyrimidine uracil leads to the formation of 1,3-dimethyluracil in good yield. The interaction of uracil and diazomethane may be expressed, therefore, as follows:



The reaction in question is applied by suspending dry, pulverized uracil in dry ether containing an excess (4–6 moles) of diazomethane and allowing the mixture to stand at ordinary temperature until the uracil completely dissolves. The reaction is complete in about 4 days when working with uracil in units of one gram. The dimethyl uracil is obtained in the form of slender colorless prisms which melt at 121.5° C. (Nitrogen determination: calculated, 20.03%; found, 19.87%.)

The study of the behavior of thymine and cytosine is now in progress. The results thus far obtained by alkylation with this reagent indicate that we may be able to use it with success for the detection of replaceable hydrogens in the more complicated pyrimidine and purine glucosides.

Where CO.NH groupings functionate in such constructions it should be possible to replace hydrogen in such amide groupings with methyl by action of this reagent. The hydrolysis of the products of alkylation should lead to the formation of methyl derivatives of known constitution, thereby contributing new knowledge regarding the constitution of the parent substances. The final results of our researches will be published in the *Journal of the American Chemical Society.*

ABSORPTION OF SHORT X-RAYS BY ALUMINIUM AND COPPER

By William Duane and K. C. Mazumder

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Communicated, February 1, 1922

1. In 1914 one of us presented to the American Roentgen Ray Society and to the American Physical Society descriptions of methods of measuring the average or "effective" wave-length of a beam of X-rays, without the use of an X-ray spectrometer. Mr. Hunt and one of us[1] have shown that even if a constant voltage (from a storage battery, for instance) drives a current through an X-ray tube the resulting X-rays are not monochromatic but contain X-rays covering a considerable range in the spectrum, that depends upon the absorption of the matter through which the rays pass, as well as upon the voltage applied. We define the term "effective" wave-length to mean the wave-length that has the same absorption in a given case as that of the whole beam. For many purposes this "effective" wave-length appears to be an important wave-length to know. The methods of estimating it described in 1914 were based upon a measurement of the absorption of the beam of X-rays in aluminium or copper and the determination of the wave-length from a curve giving the relation between the absorption of monochromatic X-rays and their wave-lengths. Curves were drawn from all the data available at the time representing this relation. It was proved that over a very wide range of wave-lengths, namely, from about 3 ångströms down to .35 of an ångström, the relation between the mass coefficients of absorption and the wave-lengths could be reasonably well represented by the equation

$$\frac{\mu}{\rho} = 14.9\,\lambda^3, \qquad (1)$$

where $\frac{\mu}{\rho}$ is the total mass coefficient of absorption and λ is the wave-length in ångströms.

Since 1914 Hull and Rice[2] and Richtmyer[3] have measured the coefficients of absorption in aluminium and copper for wave-lengths lying between .75 and .135. They find that the relation between the mass coefficient of absorption and the wave-length may be expressed approximately by an equation of the form

$$\frac{\mu}{\rho} = k\lambda^3 + s. \tag{2}$$

Hull and Rice give for the constants k and s: $k = 14.9$ and $s = .12$ for aluminium; and $k = 150$ and $s = .12$ for copper, the range of wave-lengths lying between .392 and .147.

Richtmyer gives the following values for the constants k and s: for aluminium $k = 14.5$ or 14.3 and $s = .15$ or .16; and for copper $k = 147$ and $s = .5$.

2. As a great deal of work is now being done on X-rays of wave-lengths shorter than .135, we have undertaken to extend the measurements of the absorption in aluminium and copper to cover the range of these shorter waves. We have measured the mass coefficients of absorption for aluminium and copper for wave-lengths from .165 to .095.

In making these measurements we have used a Coolidge X-ray tube with a Tungsten target, excited by the high tension, constant voltage generating plant that has been described elsewhere. The constant voltage employed for these short waves amounted to about 160,000 volts. The X-ray spectrometer employed has been described also, together with the methods of protecting it against stray radiation, etc. This protection against stray radiation becomes of very great importance for the shorter, penetrating rays. It is sufficient to call attention to the fact that the X-ray tube with its generating plant was in one room and the spectrometer in an adjoining room, and that thick lead plates were fastened to the wall between the two rooms for additional protection. Further, the entire spectrometer was in a zinc box that completely surrounded it.

3. In making measurements of the coefficients of absorption we set the spectrometer to reflect a given wave-length into its ionization chamber. We then measure the ionization current with a plate of the absorbing substance placed between the X-ray tube and the hole in the wall through which the X-rays pass to the spectrometer. We next measure the ionization current with this plate removed. From each of these currents must be subtracted the current due to scattered radiation and to the natural leak in the instrument. This correction we estimate by turning the crystal of the spectrometer into such a position that it does not reflect rays into the ionization chamber, and measuring the ionization currents both with and without the absorbing plate in place. Subtracting the correction currents from the currents obtained with the crystal in its proper position, we get the fraction of radiation passing through the absorbing substance, and from this we calculate the mass coefficient of absorption in the usual way.

Aluminium plates of density 2.70 and plates of pure copper of density 8.92 were employed.

The following table contains the data obtained:

λ	$\frac{\mu}{\rho}$	
	ALUMINIUM	COPPER
.1653	.218	1.015
.1477	.194	.745
.1381	.189	.631
.1290	.182	.538
.1124	.167	.399
.0951	.161	.299

Our data agree very well with the values obtained by Richtmyer for wave-length .135.

4. In the figure we have plotted the mass coefficients against the cubes of the wave-lengths. Points marked (+) represent our measurements. Those marked (⊙) represent Richtmyer's data.

It appears that neither curve should be regarded as strictly a straight line. Both curves bend downward in the region of the shorter wave-lengths. Owing to this curvature it is impossible to estimate with certainty the points at which the curves strike the axis of zero wave-length. Extrapolations of this kind may lead to quite erroneous conclusions. The curves, however, appear to be pointing toward the values .146 for aluminium and .13 for copper at zero wave-length.

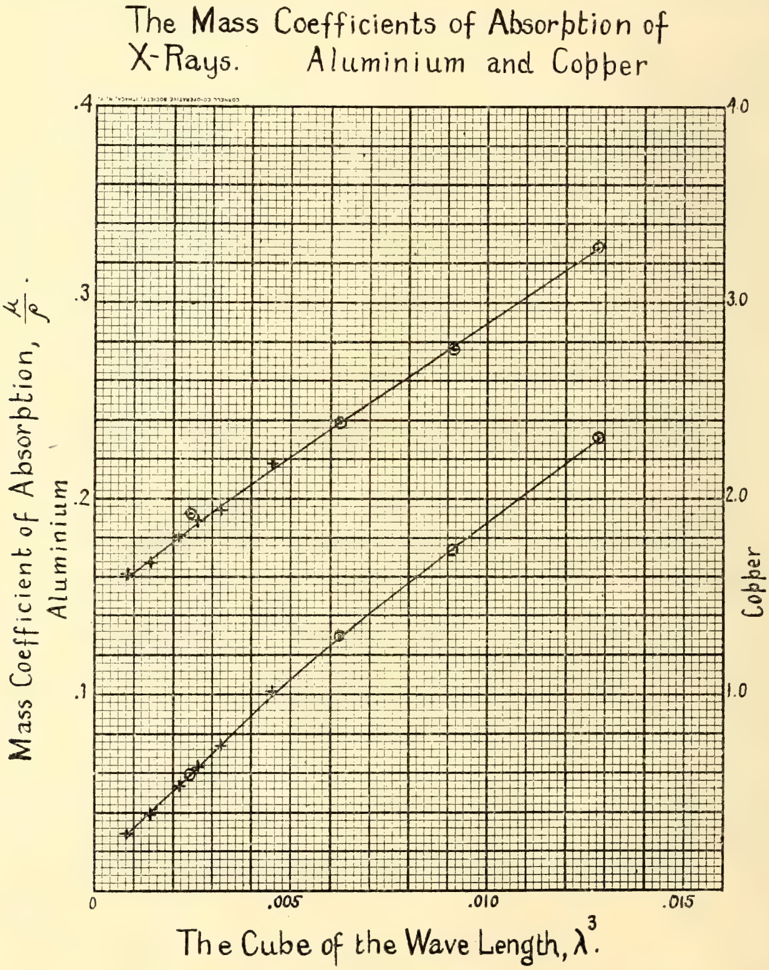
According to the usual method of analyzing coefficients of absorption into fluorescent and scattering coefficients these may be regarded as mass scattering coefficients for aluminium and copper, respectively. It is quite possible, however, that, according to some of the recent conceptions of absorption, this division of the coefficients into fluorescent and scattering coefficients may be somewhat artificial.

5. The following equations represent approximately the coefficients of absorption that we measured in the region of short wave-lengths:

$$\begin{aligned} \text{For aluminium } \frac{\mu}{\rho} &= 15.5\,\lambda^3 + .147 \\ \text{For copper } \frac{\mu}{\rho} &= 193\,\lambda^3 + .13 \end{aligned} \qquad (3)$$

It will be noticed that the mass scattering coefficient for aluminium does not differ appreciably from that obtained by Richtmyer, and that the mass scattering coefficient for copper agrees well with that given by Hull and Rice. The coefficients of the terms containing the cube of the wave-length, however, are each larger than the corresponding quantities deduced by Hull and Rice and Richtmyer from their experiments on longer waves.

6. Evidently by means of the relation between the absorption and the wave-length it becomes possible in any practical case to determine the "effective" wave-length for a beam of X-rays by simply measuring the fraction of the radiation that passes through a plate of aluminium or a plate of copper. The details of such experiments will be given elsewhere. It is sufficient to note here that owing to the rapid change of absorption with wave-length, copper should be used for estimating the wave-length of short rays, whereas aluminium is better for longer ones.



In certain practical cases (in radio-therapy, for instance) another method of estimating the "effective" wave-length appears to be preferable. This consists in determining by means of an ionization chamber, a photographic plate or a fluorescent screen the thickness of aluminium that is equivalent to a given thickness of copper. The ratio of the equivalent thicknesses of aluminium and copper changes a great deal in this short wave-length region of the X-ray spectrum. In the neighborhood of wave-length .3, for instance, an aluminium plate must be twenty-eight times as thick as a copper plate in order to absorb the same fraction of X-radiation. For wave-length .095, however, a plate of aluminium only 6 times as thick as a plate of copper will absorb the same fraction of X-radiation. The estimate of the equivalent thickness of aluminium and copper can be made comparatively easily and rapidly, so that the method furnishes a good means of determining the "effective" wave-length of a beam of X-rays during a treatment with them.

The curve giving the relation between the equivalent thickness and the wave-length will be published in a technical journal. The equivalent thickness for a given short wave-length can be calculated, of course, from equations (3), by dividing the linear coefficient of absorption for copper by that for aluminium.

The coefficient of absorption of a beam of rays that is not monochromatic usually (but not always) decreases as the rays pass through matter. If one measures the coefficient of absorption of such a beam, the results give a kind of average coefficient of absorption for the rays, as they travel through the absorbing material. Similarly, the above-mentioned methods of measuring the "effective" wave-lengths furnish a kind of average value for the "effective" wave-length of the beam as the rays pass through the plates.

[1] *Ithaca, Physic. Rev.*, August 1915, p. 166.

[2] *Ibid.*, Sept. 1916, p. 326.

[3] *Ibid.*, July 1921, p. 13.

THE EFFECTIVE TEMPERATURE OF 16 STARS AS ESTIMATED FROM THE ENERGY DISTRIBUTION IN THE COMPLETE SPECTRUM

By W. W. Coblentz

Bureau of Standards, Washington, D. C.

Communicated by V. M. Slipher, January 23, 1922

1. *Introductory Statement.*—Data on the spectral energy distribution of stars as related to that of a black body are very meager. They are the results practically of the spectrophotometric measurements of Wilsing,[1] and of Nordmann,[2] and the spectral energy curves determined photographically by Plaskett,[3] all of which relate to the visible spectrum.

The various methods used to obtain stellar temperatures give different results. For example, Plaskett (loc. cit.) obtained an effective temperature of 6800° K for γ *Cassiopeiae* (Class Bop) by considering the continuous spectrum and 10,600° K by considering the bright line spectrum.

In view of the fact that his continuous spectrum measurements terminated at 0.42 μ, where also is the observed maximum spectral energy, it is possible that the higher estimated temperature is the more nearly correct.

Wilsing, Scheiner and Münch (loc. cit.) also obtained a temperature of 6800° K for γ *Cassiopeiae.* Their temperature measurements of various stars of class *B* vary from 7000° to 15,000° K; class *A*, from 8000° to 12,000° K; class *F* from 5000 to 7000° K; class *G*, from 4000 to 5000° K; and class *M*, 3000° to 3500° K.

While it is to be expected that the various methods must give different results, it is interesting to find a rather close agreement in the estimated stellar temperatures. The agreement is especially close for stars of classes *G*, *K* and *M*, that is, stars having a low temperature.

In a previous paper,[4] data were given on a comparison of stellar radiometers and radiometric measurements of 110 stars using the Crossley reflector of the Lick Observatory at Mt. Hamilton, Calif., the altitude being about 4000 ft. Quantitative measurements were made on stars down to magnitude 5.3, and qualitative measurements to magnitude 6.7. It was found that red stars emit from 2.5 to 3 times as much *total* radiation as blue stars of the same visual magnitude.

These observations were verified by an independent method which consisted in measuring the transmission of stellar radiation through a 1-cm. cell of water, having quartz windows. By this means it was shown that of the *total* radiation emitted, blue stars have about 2 times as much *visible* radiation as yellow stars and about 3 times as much *visible* radiation as red stars.

2. *Experimental Procedure.*—In the present investigation the spectral energy distribution of a star was determined by means of a series of transmission screens, placed in front of a vacuum thermocouple which was used as the radiometer.

Screens were selected which, either singly or in combination, had a uniformly high transmission over a fairly narrow region of the spectrum terminating abruptly in complete opacity in the rest of the spectrum. By proceeding in this manner the observations required no correction other than that for surface reflection, which amounts to about 9 per cent for the two surfaces of the screen. Corrections were made for absorption by the telescope mirrors, also for atmospheric absorption, using the spectral transmission factors for the sun, as observed by Abbot and Fowle.

By means of these screens (of red and yellow glass, quartz and water) it was possible to obtain the radiation intensity in the spectrum (from the extreme ultra-violet, which is limited by atmospheric transmission and the low reflectivity of the telescope mirrors) at 0.3μ to 0.43μ; 0.43μ to 0.60μ; 0.60μ to 1.4μ; 1.4μ to 4.1μ; and 4.1μ to 10μ.

In this manner the distribution of energy in the spectra of 16 stars was determined, thus obtaining for the first time an insight into the radiation intensities in the complete spectrum of a star.

3. *Results.*—By means of this device it was found that, in the class *B* and class *A* stars, the maximum radiation intensity lies in the ultra-violet (0.3μ to 0.4μ) while in the cooler, classes *K* and *M*, stars, the maximum emission lies at 0.7μ to 0.9μ in the infra-red. From this it appears that the black body temperature (i.e., the temperature which a black body would have to attain in order to emit a similar relative spectral energy distribution) varies from 3000° C for red, class *M*, stars to 10,000° for blue, class *B*, stars.

This estimate of the effective temperature of a star was obtained by two methods. The first method consisted in making all corrections to the observations, excepting those for atmospheric absorption, and comparing them with the calculated values, using a solar type star (α Aurigae, class *Go*) as a standard. This seems permissible, in view of the fact that the observed temperature (6000° K) of α Aurigae was found to be in close agreement with that assigned to the sun. The stellar temperatures estimated in this manner are given in column 4 of table 1.

The second method consisted in applying all corrections to the observations, including the one for atmospheric absorption, and comparing the results with the calculated values. Applying factors for atmospheric absorption introduces great irregularities in the observed spectral radiation components. This, no doubt, is owing partly to selective emission of the star, and partly to the use of improper transmission factors, which, because of lack of time, could not be determined directly.

The temperature of a star, as estimated from the spectral energy components outside of the atmosphere, extends over a wide range, the average value of which is in good agreement with that obtained by the first method (see column 3 of table 1).

In table 1 are assembled the effective stellar temperatures as determined by various methods. The agreement is especially close for stars of classes *G*, *K* and *M*.

Most of the stars measured are no doubt giants, of which Russell[5] says: "The mean densities of the giant stars diminish rapidly with increasing redness, from one-tenth that of the sun for class *A* to less than one-twenty-thousandth that of the sun for class *M*." In speculating on the nature of stellar radiation and stellar temperatures it is interesting to find that the observed radiometric data (see table 2) seem conclusive in showing that the early type (class *M*) stars are losing heat 3 to 4 times as fast as the more dense, but hotter late-type (class *B*, *A*) stars. The least dense, class *M*, stars must therefore be losing heat by radiation, in which conduction cannot contribute very materially in maintaining the surface at a given temperature.

In the dense stars the shape of the spectral energy curve, and hence our inference of the effective temperature is determined by the spectral emissivity of the surface; while in the less dense stars the radiation emanates from great depths.

TABLE 1

STELLAR TEMPERATURES

STAR	CLASS	COBLENTZ		WILSING, SCHEINER & MÜNCH	NORDMANN & LEMORVAN	NORDMANN
		Calculated	Class *Go* as Standard			
ε Orionis	Bo	13000 to 14000° K	13000° K			
β Orionis	B8p	10000 to 12000	10000			
α Lyrae	Ao	8000 to 10000	8000	9400° K		12200° K
α Canis Majoris	Ao	8000 to 11000	8000			
α Cygnii	A2	8000 to 10000	9000	9400		
α Aquilae	A5	7000 to 9000	8000	8100		
α Canis Minoris	F5	5500 to 7500	6000	7200		
α Aurigae	Go	5300 to 6500	6000	7100		
α Boötis	Ko	3500 to 4500	4000	3700		
β Geminorum	Ko	4500 to 7000	5500	4900		
α Tauri	K5	2800 to 4500	3500	3500	3600	3500
α Orionis	Ma	2800 to 3300	3000	3000		
α Scorpii	Map	2500 to 3200	3000			

β Andromedae	Ma	3500 to 4500	4000	3200	4300	3700
μ Geminorum	Ma	2500 to 3300	3500	3100	3200	
β Pegasi	Mb	2500 to 3200	3000	2800		
Sun	Go		5800* to 6200			5320

* Recalculated from Abbot & Fowle, *J. Opt. Soc. Am.*, **5,** 1921 (272).

Through the generosity of Drs. V. M. Slipher and C. O. Lampland, the present data were obtained during the past summer, using the 40-inch reflector of the Lowell Observatory, at Flagstaff, Arizona, altitude 73,000 ft. Dr. Lampland personally operated the telescope, thus insuring speed and efficiency. It is a pleasure to record here my grateful acknowledgment for the many courtesies extended me by various members of the Observatory Staff.

TABLE 2

Comparison of the Total Radiation from Stars Having Closely the Same Visual Magnitude

STAR	SPECTRAL CLASS	VISUAL MAGNITUDE	GALVANOMETER DEFLECTION CM.
ε Geminorum	G5	3.18	0.58
μ Geminorum	Ma	3.19	2.11
β Geminorum	Ko	1.21	2.19
α Scorpii	Map	1.22	8.82
α Aquilae	A5	0.90	1.89
α Tauri	K5	1.06	6.03
α Orionis	Ma	0.92	15.0
ρ Ophiuchi	Ko	2.94	0.37
δ Ophiuchi	Ma	3.03	1.37
γ Leonis	Ko	2.61	0.99
β Pegasi	Mb	2.61	2.96
α Coronae Borealis	Ao	2.32	0.48
β Ursa Majoris	Ao	2.44	0.37
γ Draconis	K5	2.42	1.65
α Aurigae	Go	0.21	4.91
α Boötis	Ko	0.24	8.10

[1] Wilsing, Scheiner and Münch, *Publ. Astrophys. Obs., Potsdam*, **24,** 1920 (No. 74).

[2] Nordmann, Paris, *Compt. Rend.*, **149,** 1009 (1038). Nordmann and LeMorvan, *Ibid.*, **173,** 1921 (62).

[3] Plaskett, *Mon. Not. R. A. S.*, **80,** 1920 (771).

[4] Coblentz, *Washington, Bur. Standards, Bull.* **11,** 1914 (613).

[5] Russell, *Pop. Astron.*, **22,** Nos. 5 and 6, 1914.

ADDITIONAL STUDIES IN ARRESTED EVOLUTION

By Rudolph Ruedemann

State Museum, Albany, N. Y.

Communicated by J. M. Clarke, January 29, 1922

In a former paper on the "Palaeontology of Arrested Evolution,"[1] the writer has traced the causes of *persistence* in animals as seen from the viewpoint of the palaeontologist.

A number of conclusions as to the distribution of persistent types in the different classes and as to the probable causes of persistence were there obtained by a statistical method, in which the genera persisting through more than two geological periods were utilized. Two entirely different groups of such persistent genera were distinguished and these were termed *persistent radicles* and *persistent terminals*. The former are primitive central stocks, still adapted to a variety of conditions; the latter post-climacteric types which have become fixed in their characters by having found stable physical surroundings, or by gerontic conditions, loss of variability, etc.

In continuation of these studies the writer has investigated the possible influence recorded in fossils, of the different modes of propagation upon the persistence of types and has found that all the lower modes of propagation, viz., propagation by simple division, by budding, by her maphrodites and by parthenogenesis, are distinctly favorable to persistence, mainly through diminution in the frequency and range of variability as far as it is induced by fertilization.

In trying to trace the persistence of types to its ultimate causes, beyond the processes of heredity, ontogeny, environment and selection, around which the life and evolution of animals may be said, with H. F. Osborn. to center, it was found that the process of selection may account for the cases of persistence where variability has been reduced to a minimum by the modes of propagation mentioned before; and that environment accounts for resistance in those cases where it has become so stable as to lack the actual stimulus for further development or change of form. But it is obvious that there are still more important factors involved in heredity and ontogeny that make for persistence in organisms, especially as that is shown in the post-climacteric forms, or persistent terminals. And above all, none of the four factors, selection, environment, heredity and ontogeny, gives any clue to the actual mechanics of the processes that induce persistence in types.

Such explanation of persistent types and a clue to the mechanics of the processes involved is intimated in the views recently advanced on the methods of inheritance and production of new characters by means of the genes or character-determiners of the heredity-chromatin. Dürken and

Saalfeld[2] hold that external influences act first upon the cytoplasm (protoplasm) of the cells, and especially of the germ-cells, producing inceptive genes (plasmogenes) which finally become true genes of the heredity-chromatin when the long-continued strain has passed a certain "threshold." This final abrupt change would well account for the abrupt appearance of new characters in the ontogeny, or the salto-mutations of De Vries and the constancy of persistent types under stable external conditions. Likewise the fact that the cytoplasm is known to influence the heredity-chromatin of the nucleus, or the genes, would explain the absence of fluctomutations or variations under stable conditions. Persistent types are then either forms in which through lack of external stimulation no changes arise in the cytoplasm that produce new genes; that is, forms in which the cytoplasm is not able to react any more to external influences and differentiation could take place only through loss of genes or a combinative mode of production of new genes. This latter method leads to so-called "self-differentiation" and thereby to the excessive terminal forms of series, or to terminal persistent types, in which latter case absolute rigidity of the forms would result from the entire failure of acquisition of new genes and gerontic rigidity in the old ones.

The persistent radicles, on the other hand, correspond to the extreme development of what Saalfeld, from his studies of ammonites, terms "Konservativreihen." These are series in which the salto-mutations appear in very long intervals, while the numerous side-branches (which furnish the index-fossils) develop by rapid salto-mutations. These persistent radicles are therefore able to undergo new periods of explosive and climacteric development (Virenz-perioden of Wedekind) and are thus still less absolutely persistent than the persistent terminals.

[1] Presidential address before The Palaeontological Society, Albany meeting 1916, *N. Y. State Mus., Bull.* **196,** 1918 (107–38).

[2] *Die Phylogenes Fragestellungen zu ihrer exakten Erforschung. Berlin. Gebr. Bornträger.* 1921.

NEW FORMS OF LIFE FROM THE SILURIAN

By Rudolph Ruedemann

State Museum, Albany, N. Y.

Communicated by J. M. Clarke, January 29, 1922

The silurian of New York (Bertie waterlime and Lockport limestone) have recently yielded some extraordinary fossils which are now under investigation.

There are algae with well-preserved air-bladders, and primitive land-plants with thick coaly tissue. The graptolites, however, are the most striking: Dictyonemas, a foot and more in diameter, and varied types of the Inocaulis group. The latter possess thickly sheathed, glove-finger like basal tubes that served as anchors, with distal or upper extremities, confused masses of long, rigid or flexed thecae giving the part the appearance of a Medusa-head. By frequent division into bundles of thinner tubes, these thecae sometimes produce a still denser mass, in which the axis is a compound, spirally rising platform composed of broader thecae. In this respect the structure is identical in appearance with the graptolite genus Cactograptus.

Worms abound in certain beds of both the formations named, several entire individuals with distinct segmentation having been found, one clearly referable to Protoscolex.

Among the brachiopods a species of Lingula preserves the original concentric color-bands.

Most interesting among the cephalopods is a Hexameroceras, in which the sides of the slit leading to the hyponomic sinus are provided with interlocking teeth, serving for protection, as in the case of the narrow slit-like aperture of the living Cypraea and other gastropods. This and other observations made upon these Silurian cephalopods are significant as supporting the view lately advanced by Prell (*Centralblatt,* No. 10, 1921, pp. 303–315) that in the early nautiloids of the Phragmoceras and Gomphoceras types with variable and contracted apertures, the animals lacked all cartilaginous cephalic parts and were so plastic that they were able to squeeze their entire cephalopodium out; thus comparable to the recent Helix and Cypraea. The apertural sinuses which have served to distinguish the so-called genera Pentameroceras, Hexameroceras, etc., are therefore not as constant features as has been supposed, nor did they serve as exits for arms.

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PROCEEDINGS
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THE OXIDATION OF HYDROQUINOL IN THE PRESENCE OF ALIPHATIC AMINES

By Rolla N. Harger[1]

Department of Chemistry, Yale University

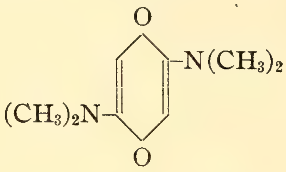
Communicated by Treat B. Johnson, March, 1922

That hydroquinol is converted by oxidation in neutral and acid solutions into quinone, with the intermediate formation of quinhydrone, was first demonstrated by Wöhler in 1844.[2] This reaction appears to proceed quantitatively and the change can be brought about by the use of a great variety of oxidizing agents.

When we turn, however, to the oxidation of hydroquinol in alkaline solution the situation is far more complicated and at present we must admit that we have not an accurate understanding of the various changes which take place. The writer's interest in this phase of the subject resulted from the observation that an aqueous solution of hydroquinol, made strongly alkaline with methylamine, rapidly absorbs oxygen gas at ordinary temperatures with formation of a brilliant purple colored compound which is characterized by its pronounced crystalline habit.

It is of course well known that alkaline solutions of hydroquinol absorb oxygen, but if one will substitute an aliphatic amine, like methylamine, for the alkali usually employed he will observe an absorption of oxygen which is unusual both for its rapidity and in regard to the quantity consumed. The characteristic reactions which take place under these conditions appear to have not been studied.

For example, if hydroquinol is dissolved in a strong aqueous solution of mono- or dimethylamine and this solution is placed in a tube and exposed to oxygen, the liquid rapidly assumes a brownish red color and within an hour the deposition of a considerable quantity of beautiful red plates is observed. At the end of twenty-four hours the solution is almost filled with these brilliant crystals and for each gram of hydroquinol used there is absorbed about 250 cc. of oxygen. If the oxidation is continued for two or three days more the total absorption will increase to 500–600 cc. of oxygen per gram of hydroquinol, or from 10 to 12 equivalents of oxygen for each molecule of hydroquinol. The colored crystalline compounds formed under such conditions are very stable and can be easily purified. The compound formed by the oxidation of hydroquinol in the presence of dimethylamine melts at 171° and is to be expressed by the formula $C_{10}H_{14}O_2N_2$. While its constitution has not yet been completely established it appears to be identical with tetramethyl diamino-quinone which was obtained by Mylius[3] by treatment of benzo-quinone with dimethylamine, and to which Kehrmann[4] has assigned the structural formula:



Aliphatic derivatives of this type are analogous in constitution to dianilino-quinone which Hofmann[5] prepared by interaction of aniline with quinone. While a large number of these anilino-quinones have been synthesized, an examination of the literature has shown that only four other representatives of these aliphatic amino quinones are known: the methylamino derivatives of toluquinone,[6] 2-5 xyloquinone,[7] and thymoquinone,[8] and the ethylamino derivative of thymoquinone.[6] These were prepared by interaction of the amine and quinone with yields which were said to be very low. The above mentioned oxidation of hydroquinol appears, therefore, of special interest as a simple, efficient means of preparing these aliphatic amino quinones, which latter compounds seem to present a chemical field as yet very little developed.

A somewhat analogous oxidation in the naphthalene series has been recorded by Plimpton.[9] He prepared amino derivatives of naphthoquinone by the action of amines upon naphthoquinone and notes that these compounds are also produced when a solution of naphthoquinol and the amine is exposed to the air. The resulting product, however, was a *mono*-aminoquinone and not a *di*-aminoquinone such as we appear to have with the benzene series.

The brilliant purple compounds corresponding to formula I are not the only products resulting from this oxidation of hydroquinol and aliphatic amines. The yield of these purple compounds never approaches quantitative values, and on concentrating the filtrate to expel the excess of free amine there is deposited, upon cooling, a considerable amount of dark yellow crystals indicating at least another compound. This latter is now being investigated. It has also been found that when an aliphatic amine is added to hydroquinol the first product of reaction is a double compound or addition product of the amine and hydroquinol. Several such combinations have already been prepared and isolated. Whether they form an intermediate step in the process of oxidation cannot be stated at present with certainty. It has also been observed that the oxidation proceeds rapidly even in air, and quite as well in alcohol as in water.

Research is now in progress dealing with the oxidation of hydroquinol, and its homologues toluquinol, xyloquinol, etc., in the presence of various aliphatic amines. The detailed results of this work will be published later in the *Journal of the American Chemical Society*.

[1] NATIONAL RESEARCH FELLOW IN CHEMISTRY.

[2] Wöhler, *Ann.*, **51**, 1844 (145–63).

[3] F. Mylius, *Ber.*, **18**, 1885 (467).

[4] F. Kehrmann, *Ibid.*, **23**, 1890 (906).

[5] A. Hofmann, *Jahr.*, 1863 (415).

[6] F. Fichter, *Ann.*, **361**, 1908 (397, 400).

[7] F. Fichter, and A. Willmann, *Ber.*, **37**, 1904 (2388).

[8] Th. Zincke, *Ibid.*, **14**, 1881 (94).

[9] R. T. Plimpton, *J. Chem. Soc.*, **37**, 1880 (637).

THE GRATING SPACE OF MICA AND THE INTENSITIES OF THE SPECTRAL ORDERS

By BERGEN DAVIS AND H. M. TERRILL

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Communicated by W. Duane, February 28, 1922

In attempting to determine the reflectivity from its principal cleavage planes, it was found desirable to know the grating space and incidentally the relative intensities of the several spectral orders.

We would like to point out that perhaps the grating space of mica may not be constant. The thickness of a piece of mica may be altered by pressure to a small extent without apparently destroying its crystalline structure. In this case we should expect nautral specimens even of the same chemical constitution to show differences due to the variable pressures to which they may have been subjected in their natural positions in the native rocks.

The specimen of mica here reported on was a clear sheet about one millimeter in thickness. It was selected for its smoothness of surface and freedom from flaws. Nevertheless it was not optically plane as shown by viewing a distant straight edge by reflection from the surface. Small local departures from a plane surface were evident.

The measurements were made by an ionization spectrometer, the standards of wave-lengths were the $K_{\alpha_1}(\lambda = .710)$ and the $K_{\beta_1}(\lambda = .633)$ emitted from a Coolidge tube having a molybdenum target. The slit of the ionization chamber was left open wide, the readings of angular position being made only on the crystal verniers.

Measurements were made on the whole spectrum, line and continuous. The relative heights of the peaks of the curves for the two lines after proper correction for the natural leak and the continuous spectrum was taken as the relative intensities of the lines in the several orders. Readings were carefully made on each side, left (L) and right (R) of the zero position of the crystal. The expression $^{1}/_{2}(\mathrm{L} - \mathrm{R} - 180^\circ)$ is then the mean grazing angle θ for the particular line and order.

The results are given in the table. The spectra of certain orders were so weak and the curve so flat that it was not possible to fix the position of the peak with sufficient accuracy to be of value in the calculation of the grating space.

The relative intensities of the α_1 and β_1 lines are given in the table. The odd orders are stronger than the even orders as observed by de Broglie. The third and fifth orders are stronger than the first. Radiation at higher orders than the 7th was observable.

The grating space d calculated for the two lines is given in the table. The values agree fairly well with those calculated by Seigbahn (*Phys. Rev.*, Sept., 1916) from the measurements made by Gorton (*Phys. Rev.*, March, 1916), for certain lines in the L radiation of tungsten.

It is noticeable that the grating space as here determined increased with increasing spectral order. The results calculated by Seigbahn from Gorton's measurements show the same tendency. This effect has also been noted by W. Stenström (Untersuchengen der Röntgen Spektra, Lund, 1919). Stenström attributes this progressive change in the apparent value of d to a very small refraction of the X-rays in the crystal. This is a possible explanation of the effect but not the only one. It may well be due to small curvature or irregularities in the crystal. One can readily show that if the reflecting planes do not line accurately at the center of rotation of the crystal, or if the beam of X-rays from the slits do not pass accurately across this center of rotation, there will be a progressive change in the calculated values of d for the various orders if the crystal planes have even a small curvature. The direction of the change of d with order will depend on whether the crystal is convex or concave.

TABLE 1

	β_1 LINE		α_1 LINE		INTENSITY	
N	θ	d	θ	d	β_1	α_1
1	1°–50′–50″	9.87	2°– 4′–0″	9.845	100	364
2	3–39–38	9.914	4– 7–7.5	9.885	68	221
3	5–3–30	9.891	6–10–15	9.901	154	544
4			8–13–15	9.931	45	175
5	9–8–38	9.957	10–16–38	9.949	136	508
6					9.1	28
7			14–27–0	9.958	32	101

IONIZATION AND RADIATION POTENTIALS AND THE SIZE OF THE ATOM

By Bergen Davis

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Communicated by W. Duane, February 28, 1922

Professor A. S. Eve has recently published an interesting note on a relation between the ionization potential and the size of the atom. (*Nature*, June 30, 1921).

For some time I had been accumulating data for a similar comparison, but from a somewhat different point of view. Eve considers that the size of the atom is determined by the radius of the outer ring. The work (ionization potential) required to lift an electron from this ring by the Bohr model is proportional to $1/a$. The product Ixa should be a constant. Trials of this relation are made for a number of elements using both the cube-roots of the atomic volumes and estimates of the diameters of atoms made by W. L. Bragg from crystal structure and measurements. (*Phil. Mag.*, Aug., 1920). The results obtained by Eve showing the degree of constancy (and departures from constancy)are repeated here in columns 6 and 7 of the table.

The boundary of the atoms, however, should not correspond exactly with the outer ring, but should extend beyond it. If the electron ring were the limit in the solid state, it would require no work to remove an electron from an atom in this state. The photo-electric effect shows that work is required, that it is less than the ionizing potential, but is approximately the same as the radiation potential. I here consider that the limit of the atom is the ring or distance from the center that an electron needs to be lifted to produce the radiation potential. That is, an atom in the solid state of matter is ionized at a potential about equal to the radiation potentials in the gaseous state. The ionizing potential is given by

$$I = A/a,$$

where A is a constant and a is the radius of the electron ring or orbit. If b is the distance from the atomic center then an electron must be lifted to produce radiation, then the radiation potential is given by

$$R = A/a - A/b,$$

where A/b is the work required to remove an electron from position b entirely from the atom in the gaseous state.

$$I - R = A/b,$$

Th product $(I - R)b$ or, for comparison with atomic volumes, $(I - R)\sqrt[3]{A.V.}$ should be a constant. The last two columns of the table show the results obtained for the above products. The values of the atomic diameters expressed in Ångstrom units are repeated from Eve's paper which were taken from a list by Bragg.

TABLE 1

E	I	R	D	$\sqrt[3]{\text{A. V.}}$	I × D	I × $\sqrt[3]{\text{A. V.}}$	(I−R) × D	(I−R) $\sqrt[3]{\text{A. V.}}$
H	13.8	10.2		2.45		33.8		8.82
				Group I				
Na	5.13	2.12	3.58	2.87	18.1	14.7	10.74	8.61
K	4.32	1.55	4.15	3.57	17.9	15.4	11.5	9.64
Rb	4.15	1.55	4.50	3.81	18.7	15.9	11.7	9.90
Cs	3.9	1.48	4.75	4.12	18.4	16.0	11.5	9.97
				Range	.8	1.3	1.00	1.37
				Group II A				
Mg	7.61	2.65	2.85	2.4	21.7	18.3	14.14	12.24
Ca	6.09	1.90	3.40	2.96	20.8	18.0	14.28	12.16
Sr	5.67	1.80	3.90	3.25	22.2	18.4	15.2	12.57
Ba	.19	1.56	4.20	3.31	21.8	17.2	15.25	12.03
				Range	1.4	1.2	1.11	.54
				Group II B				
Zn	9.35	4.02	2.65	2.09	24.77	19.5	14.12	11.2
Cd	8.95	3.79	3.20	2.35	28.64	21.0	16.25	12.12
Hg	10.38	4.85		2.45		25.4		13.47
				Range	3.87	5.9	2.13	2.27
				Group III B				
Tl	7.3	1.07	4.50	2.58	32.85	18.8	28.	16.07
				Group IV B				
Pb	7.93	1.26	3.80	2.63	30.13	20.8	25.38	17.54
				Group V A				
As	11.5	4.7	2.52	2.36	29.	27.2	17.14	16.00
				2.52		29.0		17.1
P	13.3	5.8		2.37		31.5		17.78
				2.57		34.2		19.27
				Range		4.3		1.78
						5.2		2.17
				Group VI				
S	12.2	4.78	2.05	2.5	25.	30.4	15.17	18.5
				Group VII A				
I	10.1	2.34	2.8	2.95	28.3	29.8	21.73	22.9
				Inert Gases				
He	25.6	20.5		2.86		73.		14.25
Ne	21.3	16.9	1.3	2.67	27.7	42.8	5.72	13.33
A	15.5	11.5	2.05	3.03	31.77	36.4	8.2	12.1
				Range	4.00	36.6	2.5	2.15

The values of the radiation potentials for strontium and barium have not been directly determined, but Mohler, Foote and Megers (*Sci. Paper, Bureau of Standards*, No. 403, 1920) give estimates from the wave-lengths of the tail lines of the proper spectral series for these elements. Hydrogen is added to the table, although not given by Eve. An inspection of the table brings out the following:

For Group I, the degree of constancy of the products is nearly the same, being slightly in favor of the use of I rather than (I — R). Group II A, shows, on the other hand, that (I — R) gives a better agreement than I. This is still more marked in Group II B. Group V A is considerably more favorable to (I — R). In case of the inert gases the product using I alone does not hold at all, while the product (I — R) gives nearly a constant. The product using (I — R) places hydrogen in good agreement with the first Group. Mercury, which has some of the physical qualities of the inert gases, has a product about equal to that for those gases. In case of the whole table, going from Group to Group, the products using (I — R) are much more constant than is the case with I alone.

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*PERIODS AND LOGARITHMIC DECREMENT OF THE GRAVITATION NEEDLE UNDER HIGH EXHAUSTION**

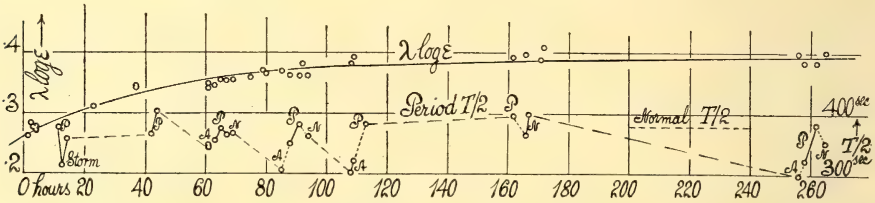
By C. Barus

Department of Physics, Brown University

Communicated, February 4, 1922

The Deflections.—After a search for a finer quartz fibre than was used in my last paper, one was found giving a double deflection of $\Delta y = 13.42 \pm .03$ cm. as compared with the former 2.67 cm., for the same case and needle. In spite of this astonishing sensitivity and otherwise admirable behavior of the apparatus, the new result for Δy from 30 successive night observations came out relatively less accurate than the former. At the same time, the observations on any single night (details must be omitted here) rarely differed by more than .1%. It is a case therefore in which the increasing importance of the radiation forces renders further finessing with the fibre of doubtful use in the given environment.

Logarithmic Decrements $\lambda \log \epsilon$.—In the endeavor to cope with this formidable difficulty, I began a study of the vibration of the needle in high vacua of a few thousandths of a millimeter; but the summer vanished before I completed it. The results so far obtained are interesting, however, and are given in the attached chart. The case was exhausted in the beginning to about 2.6×10^{-3} mm. on the McLeod gauge, and then sealed off. Air, however, in the lapse of 11 days very slowly leaked in through the glass cock and rubber tubes; and as the installment prevented me from finding the successive vacua (always beyond the U-gauge limit) I have expressed the results for T and λ in the lapse of time. To determine the latter, the needle was deflected from rest to the maximum elongation, by placing the lead balls M M' in either of the active positions; and with the needle so deflected for an instant, to quickly put the weights in the neutral position again; i.e., in the vertical plane through the needle. It is thus possible to determine both the period T of the needle and the logarithmic decrement: the former by timing successive passages (first and second) through the equilibrium position with a stopwatch, and the latter by reading the elongations. The zero point was nearly constant. The radiation is thus taken symmetrically from both sides. So far as the method goes, the values of T and λ are quite accurately found in this way, the difficulty being in the interpretation. The plenum value of $\lambda \log \epsilon$, so far as determinable in a heated room was about 1, the needle being now practically aperiodic. This has decreased to about .27 (ratio of arcs 1.9) at 2.6×10^{-3} mm. of pressure and the curve suggests a more rapid decrease beyond. With the pressure values added the curve will afford data for the viscosity of highly rarified air.



Periods.—Unlike λ, the values for the period T (figure) show no marked improvement as the vacuum increases. In spite of the steady zero point, they rather vary with the time of day, being smallest in the morning and largest toward the end of the day and they drop off at night (A denoting forenoon, P afternoon, N night). The latitude of values is enormous, ranging from about 10 minutes to over 13 minutes. The former cases (low T) are usually encountered in the morning, and probably indicate repulsive radiant forces acting from the plates toward the middle plane of the case, like an elastic buffer, whereby the period is shortened. Moreover the effect of the changes of λ on T are nearly negligible in comparison with the enormous fluctuations of T resulting from modified environmental radiation.

Whereas the deflections Δy are largest in the morning and least at night, the T values pursue a nearly opposite course, increasing from morning to afternoon, except that the night periods again drop off. When the environmental temperature is gradually increasing (morning) and when it is gradually decreasing (night), small values of T, the period of the needle, occur. This was also shown strikingly during a violent rain storm (chart 11–15 hours), in which the atmospheric temperature dropped about 20° F.

If the case is warmer, the needle meets repulsive agencies toward the center from the excess of radiation inward. If the masses m at the end of the needle are the warmer, the needle radiates toward the case outward and therefore experiences an equivalent repulsive reaction. Finally with a non-uniform radiant field of force, increasing from needle to case, the effect will be virtually an increase of the torsion coefficient, of the quartz fibre, so that T comes out smaller.

It would then follow that the large deflections are obtained when the radiant field is nearly uniform and that the larger values of the period T are the more nearly correct. This finally is in keeping with the γ-values thus obtained.

Finally the deflections Δy are modified and complicated by the screening effect of the presence of the masses M, which in the T experiments are absent.

Estimate of Radiation Temperature-differences.—Let us suppose that the radiant repulsion from case to needle, the elastic buffer effect explained, can be expressed as an increment a_r of the modulus of torsion a, of the needle. Furthermore, if f_r is the radiant force at each end and $\Delta y_r/4L$ half the double deflection in radians, we get a second value of a_r. If we equate the two values

$$f_r = \pi^2(ml/L)\,\Delta T^{-2}.\,\Delta y_r \qquad (1)$$

To find $\Delta 1/T^2$, the least and maximum T found in vacuo in the last paragraph, i.e., $T' = 5$ minutes and $T = 6.5$ minutes, may be inserted, so that the modulus in (1) is 7.0×10^{-4} nearly.

To determine the radiation forces in vacuo, it will be necessary to postulate the case of black body, or full radiation. If θ and θ_0 are the absolute temperatures between which radiation takes place and E the energy radiated (ergs per cm.^{2} per sec.)

$$E = 4K\theta^3\Delta\theta$$

The value θ may be taken as 300° and $K = 5.3 \times 10^{-4}$. Thus

$$E = 10^3 \times 5.7\,\Delta\theta.$$

This radiation may at the outset be supposed to reach the needle from a single cm.^{2} S, S', of *each* plate situated on the end normal of the needle and plate. Thus eventually, $\rho_0 = E/2\pi c$ (c being the velocity of light) is the energy density or the light pressure at a distance of 1 cm. from the element S or S'. If we suppose (as is nearly the case) the shot m of frontal area A to be situated there, the force f_r upon it will be $\rho_0 A$, where $A = .16$ cm.^{2} Hence

$$f_r = 10^{-9} \times 4.8\,\Delta\theta \qquad (2)$$

for the square centimeter of plate taken.

If now we equate the values of f_r in (1) and (2)

$$\Delta\theta = 150\,\Delta y_r \text{ nearly,} \qquad (3)$$

if but a single square cm. on each glass face radiates toward m. As there are a number of such square cms., contributing radiation, the coefficient of (3) would be decreased ten or twenty times. One may conclude I think that $\Delta\theta$ compared with Δy_r is numerically much in excess. This implies too large a $\Delta\theta$, to persist in a room of practically constant temperature, in the summer. It is thus more than probable that at a vacuum of 10^{-3} mm. or a force of .22 dynes on each shot, so small a part as 10 or 20 times the $10^{-8} \times 70 \times \Delta y_r$ instanced in equation (1) may remain fluctuating in value with small changes in the radiating thermal environment.

Measurement of γ in Terms of the Viscosity of Air.—A maximum deflection of the m-balls is produced by placing the M-balls in position. When this is attained, the latter are reversed and the velocity v with which the needle (filamentary frame) passes through the equilibrium position, is measured with a stopwatch. The M-balls are then again reversed and the measurement of v repeated; etc. In a plenum, the motion of the needle through the zero point is practically uniform and therefore the frictional force is equal to the gravitational pull. Moreover, if two points of the scale near to and equidistant from the zero point are selected for releasing and arresting the stopwatch, the torsion of the fibre acts to the same degree as an acceleration and a retardation.

The frictional resistance encountered by either round mass m of radius r is by Stokes' equation $6\pi\eta rv$, where η is the viscosity of air. The velocity v of the balls m is

$$v = (l/2L)\,\Delta y/\Delta t$$

Δy is the telescopic excursion in the time Δt, symmetrically, to the position of equilibrium. We obtain in this way

$$\gamma = \frac{R^2}{Mm}\,\frac{3\pi\eta rl}{L}\,\frac{\Delta y}{\Delta t}$$

Work on this principle has been actively pursued, but the summer closed before it could be completed.

* Advance note from a Report to the Carnegie Institution of Washington, D. C.

PLANE REFLECTION OF SOUND, AS EXHIBITED BY THE PINHOLE RESONATOR*

By C. Barus

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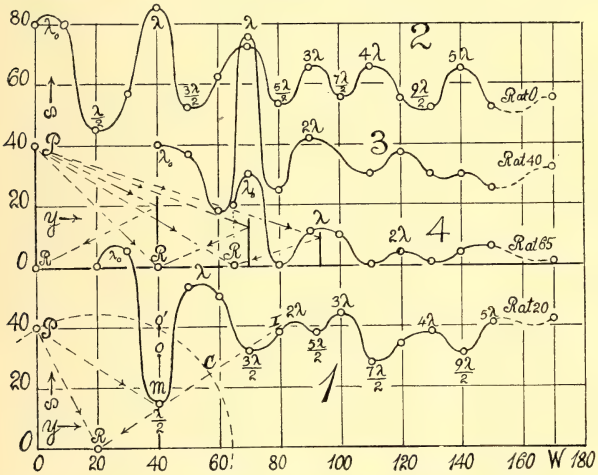
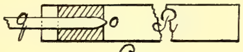
Communicated February 24, 1922

1. *Vertical Reflectors.*—These experiments, made between the walls of a room (y direction), with the closed organ pipe P, vertical, mouth down-

ward, at $z = 40$ cm. above the origin of the table, came out with surprising definiteness at once. They will therefore aid in disentangling the mysterious behavior of the resonator elsewhere observed (cf. *Science*). The resonator R, fig. 6, pin-hole tube qo, was placed on the table, horizontally, with the mouth toward the pipe (azimuth 270°). The vertical board or reflector 4 feet × 1 foot placed normal to y, was then moved successively from $y = 20$ cm. (about at the mouth of R) nearly to the wall W at $y = 174$ cm. The results are given in figure 1 where s is the fringe deflection of the U-tube interferometer. The chief maxima are near $y = 50, 100, 150$ cm., which suggests the wave-length of the f″ pipe $\lambda = 48$ cm., here operating by oblique reflection. Between the maxima λ and 3λ, there is a subsidiary maximum which one might erroneously attribute to overtones, inasmuch as it is often associated with weakly or strongly blown pipe notes. Thus at $y = 80$ the largest deflection is obtained from the dying note, at 90 from a forced note; but the 2λ crest is regular. Distances between 150 cm. and the wall W at 174 cm., were not available. A remarkable feature of the curve is the strong initial minimum at $y = 40$ cm. If the board is placed more or less obliquely to y, the minimum tends to vanish, as shown by the successive tests at o' and o.





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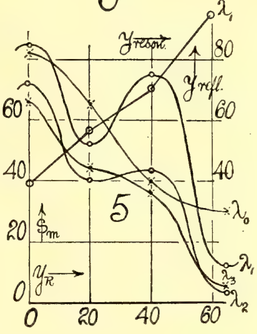

In the disposition given, two sound rays reach the resonator, one reflected from the board and the other direct, with immediate reflection from the table. When the board is near the mouth of the resonator, these rays are reduced to one and the result is an accentuated maximum. As the board distance in y increases, the path difference steadily increases with a simultaneous weakening of the reflected ray; nevertheless the figure proves conclusively that it is not as ineffective as one might suppose.

In contrast with the preceding, the next graph, figure 2, for the same pipe position, but with the resonator on the table at $y = 0$ cm. (in azimuth

180°) is more regular. The curve lies in higher values of s. There is less repetition of sectional groups. Nevertheless the same number of six maxima occur; i.e., each max. in figure 1 is reproduced and often accentuated in figure 2. The former minimum of $y = 40$ has shifted to a minimum somewhat above 20 cm.

The third survey in y was made with the pipe again at $z = 40$ cm. above the origin and with the resonator on the table (in azimuth 180°) at $y = 40$ cm. The graph, fig. 3, for the plate reflector normal to y and advancing along it shows the same number of maxima as heretofore, with the sixth soon to vanish. The feature of the curve is the enormously developed second maximum, in contrast with the first, or the others. We are no doubt trenching on the case of favorable incidence heretofore developed. The response was always stronger to the weak notes and they were used throughout, the difference being often over 10 scale parts, s.

Finally, figure 4 gives the corresponding acoustic distribution evoked by the advancing reflector with the pipe left in its former position, but with the resonator on the table at $y = 65$ cm. in azimuth 180°. The first maximum is now relatively high, while the former huge second maximum has flattened and the fifth and sixth maxima have vanished behind the wall. Weak notes were essential, as the strong notes gave relatively as little deflection, as before. As a whole, the present work adduces strikingly clear-cut evidence of the interferences to be identified in the following treatment; but it fails, as yet, to shed definite light on the anomalous distributions of intensity, s.

These maximum ordinates, s_m, for path-differences $\lambda = 0, \lambda, 2\lambda, 3\lambda$, are constructed for the corresponding resonator positions, y_r in figure 5. Compared with the regularly diminishing initial maximum, λ_0 the strongly harmonic s–distribution for the λ maximum, together with its decreasing amplitude are well brought out. Like λ_0, 2λ is but sparingly harmonic, while the odd 3λ is again markedly so. The graphs, fig 5, also contain the relation of the reflector and resonator positions in y, for λ_1 crests. The broken line should probably be straight, at an angle of about 50° to the vertical.

2. *Discussion of the Preceding Reflections.*—In the graphs 1 to 4, there are three influences at work on the resonator. In the first place the direct ray from the pipe is equivalent to a reflection from the table, a locus of reënforcement. A half wave-length has been lost. Then there are the two reflections, also with loss of $\lambda/2$, from the board normal to y and the wall W at $y = -130$ cm. This may be neglected, because it is always kept relatively very distant in the plan of the experiments. We thus have to compound the ray from the image of the source with the direct ray. The computation would be cumbersome; but the construction in the graph itself is not so and adequately correct. In figure 1, if the promi-

nent maxima or minima encountered during the march of the reflector in y, be taken, a circle is to be drawn with a center at R and a radius RP. In case of the first minimum at $y = 40$ cm., the image is I at $z = 40$, $y = 80$. Hence $IC = 27$ cm. is the path difference from the minimum at $y = 40$ cm. which is placed a little too high. The following data so measured, may be given as examples.

Minimum $y =$ 40 cm., $\lambda/2 = 27$cm. $y =$ 110 cm., $7\lambda/2 = 179$ cm.
70 cm., $3\lambda/2 = 82$ $9\lambda/2 = 220$
93 cm., $5\lambda/2 = 126$
Maxima $y =$ 53 cm., $1\lambda = 50$ cm. $y =$ 130 cm., $4\lambda = 198$ cm.
83 cm., $2\lambda = 105$ 155 cm., $5\lambda = 248$
103 cm., $3\lambda = 148$ 25 cm., $0\lambda = 4$

These results in their entirety are convincing, being quite as accurate as the location of maxima and minima admits. The minima of figure 2 treated by the same method give, for instance,

Minimum $y =$ 25 cm., $\lambda/2 = 24$ cm. $y =$ 103 cm., $7\lambda/2 = 170$ cm.
50 cm., $3\lambda/2 = 68$ 126 cm., $9\lambda/2 = 215$
80 cm., $5\lambda/2 = 125$ etc.

with similar data for the maxima.

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* Advance note from a report to the Carnegie Institution, Washington, D. C.

NOTE ON THE PROBLEM OF GREAT STELLAR DISTANCES

By Harlow Shapley

Astronomical Observatory, Harvard University

Communicated by Edwin B. Wilson, February 27, 1922

Through a study of the faint variable stars in the Small Magellanic Cloud it has been possible to check the photometric methods of measuring the parallaxes of globular clusters.

The great distances now ascribed to the globular systems require that the cluster type variables, with periods less than one day, be of high luminosity. The high absolute brightness of typical Cepheids, with periods longer than one day, is commonly accepted; but the faintness of the galactic variables of the cluster type, and their peculiarly large radial velocities, suggest that studies of their proper motions and trigonometric parallaxes will contribute little in the near future to the problem of their average absolute magnitude. It is now found, however, that the faintest variables in the Small Magellanic Cloud are Cepheids of the cluster type, and they are but one magnitude fainter than the typical Cepheids with periods of three days.

In 1904 and 1905 Miss Leavitt discovered nearly a thousand variable stars in the Small Magellanic Cloud from photographs made at Harvard's observing station at Arequipa, Peru. In addition, she found more than eight hundred variables in the Large Magellanic Cloud. Miss Leavitt undertook an investigation of the periods and light curves of these stars, but for various reasons had not progressed far at the time of her death, beyond the work reported in *Harvard Circular* 173, 1912. In that paper she gives the periods of twenty-five stars, notes that they are all of the Cepheid type, and calls attention to a remarkable relation connecting the brightness of the variables and the length of their periods. The logarithm of the period was found to increase about 0.48 for each increase of one magnitude in brightness. The longest period she found was 127 days for one of the brightest stars in the Cloud; the shortest period was 1.25 days for a variable nearly four magnitudes fainter.

In the globular clusters are found large numbers of Cepheid variable stars with periods shorter than one day, and only a few with longer periods. In my work on cluster distances I used the relation given by Miss Leavitt, first putting it on an absolute luminosity basis (following Hertzsprung), and then extending it to these cluster type Cepheids. It soon appeared that the linear formula given by Miss Leavitt did not hold, and an empirical curve was derived and used for obtaining distances of Cepheids in the clusters and throughout the galactic system in general.

Some objections have been raised to my extension of the period-luminosity curve, which identifies cluster type variables as giants rather than dwarfs. The proper motions of two of these variables are large, and many of them are in high galactic latitude. Both properties are usually, but not always, associated with low absolute magnitude.

The adopted high luminosity is strongly supported, however, by the comparable brightness in clusters of the short period Cepheids and the B type stars, as well as by the direct comparison with the few long period Cepheids in Messier 5, Omega Centauri, and other globular clusters. Spectrophotometric analyses of the cluster stars afford additional support. A direct extension of the observed period-luminosity curve to cluster type variables in the Small Magellanic Cloud itself would appear to be especially decisive.

Most of the known variable stars in the Cloud have median magnitudes corresponding to periods between two and five days. A few are fainter. About twenty-five of these faint variables have been investigated during the past year on a special series of long-exposure photographs made for

Variable	Period *d.*	Median	Variable	Period *d.*	Median
1408	0.715	16.0	1823	0.795	16.0
1420	0.67*	16.4	1928	0.608	15.8
1634	0.81	15.6	1943	0.81*	16.2
1655	0.444:	16.2	1964	0.687*	16.1
1731	0.65	16.2	2002	0.702	16.4
1739	0.63*	16.4	3610	0.6	16.2
1741	0.400:	16.2			

the purpose at Arequipa. Further observation is necessary before definitive results can be given, but for the preceding variables provisional periods and median magnitudes have been obtaained.

The numbers, except the last, are those published by Miss Leavitt. Colons indicate periods not well determined. For stars marked with an asterisk a multiple of the period given above satisfies the observations nearly or quite as well.

The light curves are in all cases of the typical cluster form. From the plates now available, the elements of variation could not be determined for some of the other faint stars, but for none do the observations demand a long period.

The median magnitudes are derived from published values of maximum and minimum, except that for No. 1741 which is a revised unpublished value derived by Miss Leavitt, and that for No. 3610—a hitherto unannounced variable. The medians are based on a provisional and apparently homogeneous system of magnitudes for the comparison stars. The standardization of these magnitudes is under way and will permit a new evaluation of the medians for the variable stars and a redetermination of the distance of the Small Magellanic Cloud.

The mean of the thirteen periods is 15.4 hours. The mean of the thirteen median magnitudes is 16.13 ± 0.05, in close agreement with the median magnitude, 16.2, predicted from the extended period-luminosity curve for variables of this period. We are led to the conclusion, even after making ample allowance for the provisional nature of the magnitude scale and for the uncertainties that affect some of the periods, that cluster type variables are giant stars in the Small Magellanic Cloud, as previously found for certain globular clusters. Naturally we infer that they are giants stars wherever they occur.

THE VITAL INDEX OF THE POPULATION OF ENGLAND AND WALES, 1838–1920[1]

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The senior author[2] has suggested the term "vital index" as a convenient designation of the function

$$V = \frac{100 \ Births}{Deaths}$$

which measures more effectively than any other demographic function yet devised the essential biological fitness of a population, in the sense of organic evolution. For a population V is the direct measure of survival value. If $V > 100$ the population is not only surviving but growing, whereas if $V < 100$ the population is starting on the road to elimination, and must eventually disappear unless the vital index gets above 100. A detailed presentation of the facts regarding the vital index of certain elements of the American population has lately been published.[3]

In examining the effect of the Great War upon the growth of population one cannot fail to be impressed by the fact that the return of the birth/death ratio (vital index) to a normal or even super-normal value after the marked dislocation produced by the war, and at its end the superimposed influenza pandemic, was extraordinarily rapid. Indeed the senior author[4] was led to remark, after examining the matter in the chief belligerent countries, that "Altogether, these examples, which include the effects of the most destructive war known to modern man, and the most devastating epidemic

TABLE 1

RATIO V, OF 100 BIRTHS TO DEATHS IN ENGLAND AND WALES IN EACH QUARTER OF THE YEARS 1838 TO 1920, INCLUSIVE

	QUARTERS ENDED					QUARTERS ENDED					QUARTERS ENDED			
YEAR	MAR.	JUNE	SEPT.	DEC.	YEAR	MAR.	JUNE	SEPT.	DEC.	YEAR	MAR.	JUNE	SEPT	DEC.
1838	116	134	157	140	1866	142	150	153	158	1894	155	186	201	180
1839	138	146	157	141	1867	145	178	176	158	1895	141	179	174	161
1840	134	143	148	135	1868	166	184	147	160	1896	163	185	181	168
1841	135	151	164	150	1869	153	159	166	149	1897	164	179	169	170
1842	141	155	150	148	1870	143	168	155	151	1898	154	183	166	169
1843	144	150	167	150	1871	151	167	159	144	1899	157	182	151	152
1844	142	160	163	143	1872	155	173	170	176	1900	132	166	175	167
1845	137	153	177	163	1873	163	174	178	160	1901	158	182	169	167
1846	162	166	136	128	1874	157	176	169	150	1902	153	179	209	168
1847	122	130	136	123	1875	132	164	174	159	1903	170	195	207	169
1848	116	150	160	144	1976	162	179	180	177	1904	157	192	178	165
1849	145	150	100	139	1877	171	170	195	177	1905	160	192	195	172
1850	147	168	171	159	1878	158	177	172	155	1906	169	189	181	166
1851	149	160	165	150	1879	145	167	210	160	1907	143	187	216	161
1852	152	158	151	152	1780	152	186	167	165	1908	153	201	208	170
1853	137	147	160	140	1881	163	186	196	177	1909	143	189	218	172
1854	144	168	136	133	1882	159	182	186	163	1910	163	203	218	168
1855	123	155	177	153	1883	158	171	189	167	1911	156	188	158	169
1856	164	173	173	164	1884	170	181	167	165	1912	154	188	216	170
1857	157	170	160	146	1885	157	168	191	173	1913	148	187	199	172
1858	136	158	160	133	1886	147	184	179	168	1914	149	182	197	159
1859	144	166	161	155	1887	153	176	178	163	1915	125	154	180	134
1860	149	157	190	158	1888	149	179	199	169	1916	136	165	193	134
1861	143	172	170	159	1889	158	183	180	165	1917	109	136	172	136
1862	149	173	187	150	1890	136	171	180	139	1918	118	144	147	67
1863	145	160	154	155	1891	145	139	192	156	1919	75	137	197	195
1864	135	161	161	144	1892	120	176	197	168	1920	197	208	245	180
1865	138	167	160	148	1893	160	177	160	147					

TABLE 2

GROUPED DATA FOR VITAL INDEX AND CRUDE BIRTH RATE

PERIOD	BIRTH	DEATHS	100 BIRTHS / DEATHS	GENERAL BIRTH RATE PER 1000
1838–1839	956,361	681,744	140.28	31.0
1840–1844	2,600,288	1,756,431	148.04	32.2
1845–1849	2,797,329	2,003,657	139.61	32.6
1850–1854	3,080,095	2,030,528	151.69	33.9
1855–1859	3,300,929	2,126,461	155.23	34.3
1860–1864	3,560,830	2,263,769	157.30	34.9
1865–1869	3,830,527	2,438,121	157.11	35.3
1870–1874	4,100,856	2,541,625	161.35	35.5
1875–1879	4,399,070	2,623,391	167.69	35.6
1880–1884	4,451,771	2,591,038	171.81	33.8
1885–1889	4,450,173	2,620,108	169.85	32.0
1890–1894	4,486,912	2,778,642	161.48	30.5
1895–1899	4,611,116	2,771,151	166.40	29.6
1900–1904	4,691,038	2,739,365	171.25	28.5
1905–1909	4,637,271	2,613,992	177.40	26.7
1910–1914	4,411,823	2,519,713	175.09	24.2
1915–1919	3,623,894	2,685,456	134.95	20.8
1920	957,994	466,213	205.48	25.5

since the Middle Ages, furnish a substantial demonstration of the fact that population growth is a highly self-regulated biological phenomenon. Those persons who see in war and pestilence any absolute solution of the world problem of population, as postulated by Malthus, are optimists indeed. As a matter of fact, all history definitely tells us, and recent history fairly shouts in its emphasis, that such events make the merest ephemeral flicker in the steady onward march of population growth."

In view of the fact that this function *V* has so quickly returned to normal, or approximately normal, values after the recent disturbances it is a matter of considerable interest and significance to the student of problems of public health in the broad sense, of population growth, and of human ecology, to see how precisely the self-regulatory mechanism of populations operates over long periods of time. To this end we have examined all the existing continuous statistics of England and Wales, and desire to present the results in this paper.

Table I gives the vital index (100 births/deaths) for the population of England and Wales, in each quarters of the years 1838 to 1920 inclusive. The data from which the computations were made were taken from official sources.[5]

The immediately striking feature of this table is the extreme constancy of the values as one runs down the columns. In order to appreciate this fact fully, however, it is necessary to resort to graphical presentation. As the data of table I are too detailed to make such an examination of the trend feasible within the space of a single printed page, the data have been combined in quinquennial periods in table 2. In this table are given, for each 5 year period, the total births, total deaths, vital index, and crude birth rate. The latter figures have been calculated directly from the birth and population figures and differ slightly (but in no wise significantly) from the crude birth rates given in Annual Reports of the Registrar-General, chiefly because of the fact that we have used a different quinquennial grouping.

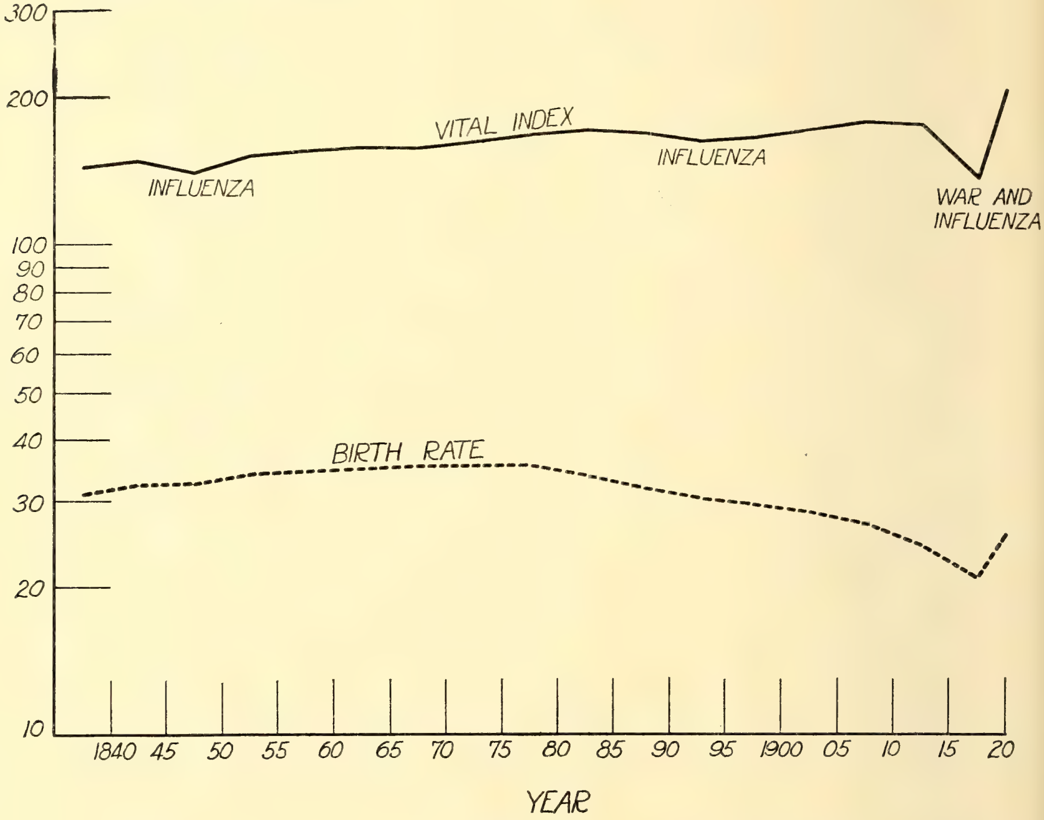

FIG. 1

Trend of vital index (100 births/deaths) and crude birth rate in England and Wales, 1838–1920, inclusive

The vital index and birth rate data from table 2 are shown graphically in figure 1.

The diagram is plotted on an arithlog grid, in order that the slopes of the lines may be strictly comparable.[6]

It is at once apparent that *the ratio of births to deaths in England and Wales had a slow but extremely even and steady increase from 1838 to 1914.* This steady progress was interrupted to a degree sufficient to be apparent upon only two occasions during the three quarters of a century. These were in 1847–1849 and 1890–1892. These fluctuations, which only slightly affected the even upward trend of the curve were due to the influenza pandemics of 1847–8 and 1890–1891. The broad result is perfectly clear and outstanding. The population of England and Wales is today biologically fitter and possessed of greater purely biological survival value as a whole population than it was three quarters of a century ago. Whether it is a mentally, morally or anthropmetrically fitter population does not now concern us. We are dealing at the moment solely with the fact that, taking the people of England and Wales as a whole, slightly over 2 babies were born for every death in a year in 1920, as against 1.4 babies per death in a year in 1838–1839. The question of a differential fertility among different social classes we shall discuss farther on in the paper.

Now this result, which is beyond any question a fact, will strike anyone informed as to the sociological and eugenical literature of the last two decades as curiously at variance with the pessimistic tenor of that literature, taken as a whole. It has been pronounced from high places that the general trend of the British people was biologically downwards, that they were in fact becoming biologically dangerously near to a decadent race. Abundant quotations in support of this contention could be cited, were space available and were it necessary. This gloomy view has had its foundation solely upon the fact that, since the quinquennium 1875–1880, the birth rate in England and Wales has been falling rather rapidly, as is clearly shown in figure 1. This fact has been brought out by Elderton[7] in great detail.

But from a purely biological view-point, what matters a falling birth rate if the death rate falls even more rapidly, so that the net survivorship at any instant of time is constantly getting higher? To this it will, of course, be answered at once by those who view with alarm the declining birth rate that the real crux of the matter is in the differential change in fertility. Nowadays the "best" people do not produce their due share of progeny, while the "worse" people over-produce. If one accepts the man-made definitions of "best" and "worst" this is plainly true for some civilizations at least. In the American population, however, as one of us[3] has recently shown, the element probably least desirable racially, the negro, has the lowest survival value as a group (vital index generally less than 100). But is old Dame Nature ever really interested in any test of "best" and "worst" *except survival?* There is no evidence as yet that she is. Measured by this test the population as a whole of England and Wales is today biologically more vigorous than it was in 1838. This is a plain *fact.* Whether this fact is, sociologically or eugenically considered, a bad thing or a good thing, is something more than a purely biological problem, and as such we are content to leave it to the discussion of others.

What seems to us of considerable interest biologically, is the extraordinary perfection, demonstrated in the figures here presented, of the self-regulation of population growth. Disturbances of the course of the birth rate have been compensated for, in the population and period under review, with the greatest nicety. It will be of interest to examine the facts for other populations, and this we hope later to do.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 50.

[2] R. Pearl, "The Effect of the War on the Chief Factors of Population Change," *Science* (N. S.), **51,** 1920 (553–556).

[3] Pearl, "The Vitality of the Peoples of America," *Amer. J. Hyg.*, **1,** 1921, pp. 592–674.

[4] Pearl, "A Further Note on War and Population," *Science* (N. S.), **53,** 1921, pp. 120–121.

[5] *Quarterly Return of Marriages, Births, and Deaths Registered in England and Wales, etc.* Published by Authority of the Registrar-General. London. H. M. Stationery Office. *Passim.* Also corresponding Weekly Return, and Annual Reports.

[6] " For a Discussion of the Significance and Advantages of Plotting Trend Lines on a Logarithmic Scale," cf. Fisher, I. " The 'ratio' Chart for Plotting Statistics," *Q. P. Am. Stat. Ass.*, **15,** 1917, pp. 577–601, and Field, J. A., "Some Advantages of the Logarithmic Scale in Statistical Diagrams," *J. Pol. Econ.*, **25,** 1917, pp. 805–841.

[7] E. M. Elderton, "Report on the English Birth Rate. Part I. England North of the Humber." *Eugenics Lab. Mem.*, **19** and **20,** Cambridge Univ. Press, 1914, pp. viii and 246.

SEASONAL FLUCTUATIONS OF THE VITAL INDEX OF A POPULATION[1]

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In earlier papers the writer[2] has discussed various aspects of the birth-death ratio, or vital index of a population, including age and sex specific vital indices, racial and secular fluctuations, etc. Table 1 of the preceding paper by Pearl and Burger[3] gives the necessary data for finding, for one population, that of England and Wales, the course of the seasonal fluctuation of this important demographic constant. It is the purpose of this note to set forth biometrically the facts on this point, from the basic data mentioned.

Table 1 gives the frequency distributions for variation in the value of the vital index of the population of England and Wales in each of the four quarters of the year, ending, respectively, on March 31, June 30, September 30, and December 31, during the years 1838 to 1920, inclusive. The source of the data is given in Pearl and Burger already cited.

The significant biometric constants from table 1 are presented in table 2. It is apparent that there are a few highly aberrant observations in

TABLE 1

Frequency Distribution for Variation in the Vital Index in the Four Quarters of the Year

VITAL INDEX	QUARTERS ENDING			
	MARCH 31	JUNE 30	SEPT. 30	DEC. 31
65– 69	—	—	—	—
70– 74	—	—	—	—
75– 79	1	—	—	—
80– 84	—	—	—	—
85– 89	—	—	—	—
90– 94	—	—	—	—
95– 99	—	—	—	—
100–104	—	—	1	—
105–109	1	—	—	—
110–114	—	—	—	—
115–119	3	—	—	—
120–124	3	—	—	1
125–129	1	—	—	1
130–134	3	2	—	4
135–139	9	3	3	4
140–144	11	2	—	7
145–149	13	2	3	5
150–154	9	7	5	9
155–159	13	6	5	11
160–164	10	5	10	9
165–169	2	11	9	17
170–174	3	9	8	7
175–179	—	11	8	4
180–184	—	9	6	2
185–189	—	10	3	—
190–194	—	2	4	—
195–199	1	1	8	1
200–204	—	2	1	—
205–209	—	1	3	—
210–214	—	—	1	—
215–219	—	—	4	—
245–249	—	—	1	—
Total	83	83	83	83

TABLE 2

Biometric Constants Deduced from Table 1

QUARTER ENDING	MEAN	STANDARD DEVIATION
March 31	147.06 ± 1.21	16.32 ± .85
June 30	169.89 ± 1.21	16.35 ± .86
September 30	175.43 ± 1.66	22.39 ± 1.17
December 31	156.46 ± 1.27	17.14 ± .90

the series, but it has not been thought best to discard or adjust them in this first treatment of the subject. Consequently it is to be understood that all the values given in table 1 were used in computing the values in table 2.

From these data the following points are to be noted:

1. The vital index has the lowest mean value in the quarter ending on March 31, the winter quarter. In that period the birth incidence is relatively low and the death incidence relatively high.

2. Next in value to this, but standing 9.40 ± 1.75 points above it, is the mean vital index for the autumn quarter ending December 31. The difference being 5.4 times its probable error, may be regarded as significant.

3. The spring quarter, ending June 30, shows the next higher mean, being 13.43 ± 1.75 points above the winter quarter.

4. The highest value of the index falls in the summer quarter, when births are most frequent and deaths least so. The mean value, however, lies only 5.54 ± 2.05 above that for the spring quarter, a difference which cannot be regarded as significant.

5. In variability of the vital index, the first two and the last quarters of the year, all exhibit significantly the same status. The vital index is distinctively more variable in the summer quarter, the difference in standard deviations when this quarter is compared with that ending June 30, amounting to 6.04 ± 1.45. This may be regarded as significant, being 4.2 times its probable error.

It thus appears that the extremely close compensatory relation between birth rate and death rate, which Pearl and Burger[3] have shown to hold in annual figures, does not obtain within the single year. Instead there is a well-marked statistically significant intra-annual, or seasonal fluctuation of the birth-death ratio.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 51.

[2] Cf. particularly Pearl, R., "The Vitality of the Peoples of America," *Amer. J. Hyg.*, **1,** 1921 (592–674).

[3] Pearl, R. and Burger, M. H., "The Vital Index of the Population of England and Wales, 1838–1920," *Proc. Nat. Acad. Sci.*, **8**, 1922, pp. 71–76.

A SOLUTION OF THE LINEAR MATRIX EQUATION BY DOUBLE MULTIPLICATION

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1. *Methods of Sylvester and of Maclagan Wedderburn.*—A matrix A of order N is defined as a binary assemblage of N^2 elements a_{ik}, where either subscript may have any value from 1 to N. We add two matrices by adding corresponding elements. We multiply by the rule

$$(AB)_{ik} = \Sigma a_{is} b_{sk};\ (s = 1, 2, \ldots, N). \qquad (1)$$

If $A_1, A_2, \ldots, A_h$ and $B_1, B_2, \ldots, B_h$ and C are all known matrices, while x is a required matrix, the linear matrix equation may be written

$$\theta(x) = A_1xB_1 + A_2xB_2 + \ldots + A_hxB_h = C. \qquad (2)$$

In 1884 Sylvester attacked this equation with characteristic energy.[1] He pointed out that, since each element of the matrix $\theta(x)$ is a linear function of the elements of x—the equation (2) being thus equivalent to N^2 linear equations of ordinary algebra—it follows θ has the properties of a matrix of order N^2 and satisfies a Hamilton-Cayley equation

$$\theta^n - m_1\theta^{n-1} + m_2\theta^{n-2} - \ldots + (-1)^n m_n I = 0 \qquad (3)$$

where $n = N^2$ and where I denotes the matrix of order N^2 whose elements I_{ik} are unity when subscripts are equal, otherwise zero. If we know the coefficients $m_1 \ldots m_n$ we can solve (2), as is evident on multiplying (2) and (3) through by θ^{-1}. We can find all the coefficients if we know the rule for finding m_n which is the determinant of the matrix θ; for the left side of (3) may be found by forming the determinant of $gI - \theta$ and afterwards putting θ in place of g. *To Sylvester, solution of the equation (2) was equivalent to finding, in terms of the matrices $A_1 \ldots A_h$ and $B_1 \ldots B_h$ the determinant of order N^2 belonging to the operation θ.*

His general method consists in setting up the two matrices of order N and of extent (étendue) h

$$x_1A_1 + x_2A_2 + \ldots + x_hA_h, \; x_1B_1 + x_2B_2 + \ldots + x_hB_h$$

where x_1, x_2, etc. are scalar variables. By forming the determinants of these two matrices we have two quantics in the variables. He shows that the required determinant is a *contrariant* called the *nivellant* of these two quantics, and is an integral but not necessarily a rational function of the coefficients of the quantics. These coefficients are regarded as known quantities. The formation of the contrariant is in general difficult. He works out the solution for several special cases. With regard to the general case he remarks[2]

"Pour présenter l'expression générale de ce déterminant pour une matrice d'un ordre et d'une étendue quelconques, c'est-à-dire pour résoudre l'équation linéaire en matrices dans toute sa généralité, il faudrait avoir une connaissance des propriétés des formes qui va beaucoup au delà des limites des facultés humaines, telles qu'elles se sont manifestées jusqu'au temps actuel, et qui, dans mon jugement, ne peut appartenir qu'à l'intelligence suprême."

But the indeterminate product of Gibbs was at that time unborn, or at least very young.

In 1904 Maclagan Wedderburn[3] gave a solution of (2) by infinite series. His method consists essentially in selecting one of the known matrices, as A_j, multiplying *by* A_j^{-1} and *into* B_j^{-1} giving (2) in the form

$$x + \psi(x) = A_j^{-1} C B_j^{-1} = C'$$

where ψ is similar in form to θ but contains h-1 terms. Then

$$x = (1 + \psi)^{-1} C' = (1 - \psi + \psi^2 \ldots)C'.$$

The choice of the term $A_j x B_j$ to be freed from matrix coefficients affects the convergence of the solution.

2. *Transformation from Matrices to Double Dyadics.*—By Gibbs' double multiplication we may set up explicit formulas for the coefficients $m_1 \ldots m_n$ of the Hamilton-Cayley equation (3). It is known that any matrix A is equivalent to a dyadic $\Sigma\mathbf{aa}'$ where $\mathbf{a}$ and $\mathbf{a}'$ are vectors. Any matrix term of the form AxB is equivalent to a sum of terms of the form $\mathbf{aa}'.\mathbf{X}.\mathbf{bb}'$ where $\mathbf{X}$ is the dyadic equivalent to x. We now introduce the double dot product defined by the rule $\mathbf{ab}:\mathbf{xy} = \mathbf{a}.\mathbf{xb}.\mathbf{y}$, and may write

$$\mathbf{aa}'.\mathbf{X}.\mathbf{bb}' = \mathbf{ab}'\mathbf{a}'\mathbf{b}:\mathbf{X}. \qquad (4)$$

Any term AxB is therefore equivalent to a sum of terms of the form $\mathbf{MN}:\mathbf{X}$ where $\mathbf{M}$ and $\mathbf{N}$ are dyadics, and the linear matrix equation (2) is equivalent to

$$\varphi:\mathbf{X} = \Sigma\mathbf{MN}:\mathbf{X} = \mathbf{C} \qquad (5)$$

where φ is a dyadic whose antecedents and consequents are dyadics, say a *double dyadic*.

3. *Dyadics Treated Like Vectors in Space of Higher Dimensions.*—Adopting unit vectors $\mathbf{e}_1 \ldots \mathbf{e}_n$ such that $\mathbf{e}_i.\mathbf{e}_k$ is unity when subscripts are equal, otherwise zero, any dyadic $\mathbf{M}$ may be expanded in terms of the dyads $\mathbf{e}_i\mathbf{e}_k$; and if we agree on some definite order, no matter what, among these dyads we may set $\mathbf{E}_r = \mathbf{e}_i\mathbf{e}_k$, *lettering r run from 1 to N^2 as i and k each run from 1 to N.* With $n = N^2$, any dyadic $\mathbf{M}_t$ may be expanded

$$\mathbf{M}_t = \mathbf{E}_1 m_{t1} + \mathbf{E}_2 m_{t2} + \ldots + \mathbf{E}_n m_{tn} \qquad (6)$$

where m_{t1} etc. are scalars. The double dyadic φ may be expanded as

$$\varphi = \mathbf{E}_1\mathbf{M}_1 + \mathbf{E}_2\mathbf{M}_2 + \ldots + \mathbf{E}_n\mathbf{M}_n. \qquad (7)$$

Now $\mathbf{e}_i\mathbf{e}_k:\mathbf{e}_r\mathbf{e}_s$ is unity when *both $i = r$ and $k = s$*, otherwise zero. This is the same assaying $\mathbf{E}_p:\mathbf{E}_q$ is unity when subscripts are equal, otherwise zero. It is immediately evident that the double dyadic φ will behave with reference to double dot product as would an ordinary dyadic in n- dimensional space with reference to ordinary dot product, and equally evident that the whole of the usual matrix theory will apply to φ with no essential change.

4. *Formation of the Hamilton-Cayley Equation.*—Since $\varphi:\mathbf{X} = \theta(x)$ the Hamilton-Cayley equation for φ will be the same as that for θ. It remains to show how to calculate the coefficients $m_1 \ldots m_n$ of this equation. Let $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_p$ and $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_p$ be any dyadics:

Definition.—The scalar $((\mathbf{M}_1\mathbf{N}_1, \mathbf{M}_2\mathbf{N}_2, \ldots, \mathbf{M}_p\mathbf{N}_p))_S$ is defined to be the sum of terms computed as follows: the leading term is the product of double dot products $\mathbf{M}_1:\mathbf{N}_1\mathbf{M}_2:\mathbf{N}_2 \ldots\ldots\ldots \mathbf{M}_p:\mathbf{N}_p$; the other terms are of like form and are obtained from the leading term by keeping the antecedents $\mathbf{M}_1 \ldots\ldots \mathbf{M}_p$ fixed in position while the consequents $\mathbf{N}_1 \ldots \mathbf{N}_p$ are permuted in all ways. The sign of any term is plus or minus according as the number of simple interchanges needed to form that term is even or odd.

Thus if $p = 2$ we shall have

$$((\mathbf{M}_1\mathbf{N}_1,\ \mathbf{M}_2\mathbf{N}_2))_S = \mathbf{M}_1 \colon \mathbf{N}_1\mathbf{M}_2 \colon \mathbf{N}_2 - \mathbf{M}_1 \colon \mathbf{N}_2\mathbf{M}_2 \colon \mathbf{N}_1.$$

If $\varphi_1, \varphi_2, \ldots, \varphi_p$ are double dyadics defined as in (7) the scalar $((\varphi_1, \varphi_2, \ldots, \varphi_p))_S$ is defined to be the result of expanding each φ as in (7), multiplying out term by term, and adding the scalars of all the terms of the product.

If each **M** be expanded as in (6) the scalar elements of every φ will form a square matrix m_{ts} of order N^2 or n, *so that each* φ *depends on* N^4 *scalar elements.* The scalar $((\varphi_1, \varphi_2, \ldots, \varphi_p))_S$ is the sum of cubic determinants of order p whose main diagonals lie on the main diagonals of the respective matrices m_{ts}; the p matrices m_{ts} are to be regarded as p nonsignant layers of a cubic matrix.[4]

We may now suppose these p double dyadics to be all equal. The scalar just written becomes $((\varphi, \varphi, \ldots$ to p factors$))_S$ and may be abbreviated $((\varphi^p))_S$.

Theorem.—The scalar $((\varphi^p))_S = p!m_p$ where m_p is the coefficient of φ^{n-p} in the Hamilton-Cayley equation for φ.

Proof. Let φ be expanded as in (7). The terms of the indeterminate product $(\varphi, \varphi, \ldots$ to p factors) are of the form

$$(\mathbf{E}_i\mathbf{M}_i,\ \mathbf{E}_j\mathbf{M}_j,\ \mathbf{E}_k\mathbf{M}_k, \ldots \mathbf{E}_r\mathbf{M}_r) \qquad (8)$$

where any subscript will have any value from 1 to n. Since $\mathbf{E}_s \colon \mathbf{M}_t = m_{ts}$ the scalar of the expression (8) has for its leading term $m_{ii}m_{jj}m_{kk} \ldots m_{rr}$ and, by the definition, the scalar is the determinant of order p of which this is the leading term, a minor of the determinant of φ. Thus the scalars of all expressions (8) vanish except when the subscripts $i, j, k, \ldots, r$ are all different. Each choice of subscripts $i, j, k, \ldots, r$ will occur $p!$ times. The scalar $((\varphi^p))_S$ is therefore $p!$ times the sum of minors of order p taken along the main diagonal of the determinant of φ. By the usual matrix theory then $((\varphi^p))_S = p!m_p$ as was to be proved.[5]

5. *Transformation Back to Matrices.*—It has thus been shown how the invariants $m_1 \ldots m_n$ of the Hamilton-Cayley equation for φ or θ may be calculated by products which are closely analogous with Gibbs' products of ordinary dyadics. To complete the solution we need to see how these invariants may be formed directly from the given matrices $A_1 \ldots A_h$ and $B_1 \ldots B_h$ without the need of first forming φ by the rule (4).

Consider first m_1, the coefficient of φ^{n-1}. If $\varphi = \Sigma\mathbf{MN}$ we have $m_1 = \Sigma\mathbf{M}\colon\mathbf{N} = \Sigma\mathbf{ab}'\colon\mathbf{a}'\mathbf{b} = \Sigma\mathbf{a}.\mathbf{a}'\mathbf{b}.\mathbf{b}' = \Sigma A_S B_S$ where A_S and B_S are the scalars of the dyadics corresponding to A and B, or, in matrix language, they are the respective sums of the elements of the main diagonals. Thus

$$m_1 = A_{1S}B_{1S} + A_{2S}B_{2S} + \ldots + A_{hS}B_{hS}. \qquad (9)$$

For m_2 we have, by the distributive character of all the steps involved in the process,

$$\begin{aligned} 2m_2 &= ((\varphi, \varphi))_S = \Sigma((\mathbf{M}_i\mathbf{N}_i, \mathbf{M}_k\mathbf{N}_k))_S \\ &= \Sigma((\mathbf{a}_i\mathbf{b'}_i\mathbf{a'}_i\mathbf{b}_i, \mathbf{a}_k\mathbf{b'}_k\mathbf{a'}_k\mathbf{b}_k))_S \\ &= \Sigma[(\mathbf{a}_i\mathbf{b'}_i{:}\mathbf{a'}_i\mathbf{b}_i)\ (\mathbf{a}_k\mathbf{b'}_k{:}\ \mathbf{a'}_k\mathbf{b}_k) - (\mathbf{a}_i\mathbf{b'}_i{:}\mathbf{a'}_k\ \mathbf{b}_k)\ (\mathbf{a}_k\mathbf{b'}_k{:}\mathbf{a'}_i\mathbf{b}_i)] \\ &= \Sigma[\mathbf{a}_i.\mathbf{a'}_i\mathbf{b}_i{:}\ \mathbf{b'}_i\mathbf{a}_k.\mathbf{a'}_k\mathbf{b}_k.\mathbf{b'}_k - \mathbf{a}_i.\mathbf{a'}_k\mathbf{a}_k.\mathbf{a'}_i\mathbf{b}_k.\mathbf{b'}_i\mathbf{b}_i.\mathbf{b'}_k] \\ &= \Sigma[A_{iS}B_{iS}A_{kS}B_{kS} - (A_iA_k)_S(B_kB_i)_S] \qquad (10) \end{aligned}$$

where, as before, A_{iS} is the sum of elements of the main diagonal of A_i, etc. and $(A_iA_k)_S$ is the sum of elements of the main diagonal of the product of the two matrices A_i and A_k.

By similar transformation we may show that

$$\begin{aligned} 6m_3 &= \Sigma[A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS} - A_{iS}B_{iS}(A_jA_k)_S(B_kB_j)_S \\ &\quad - A_{jS}B_{jS}(A_kA_i)_S(B_iB_k)_S - A_{kS}B_{kS}(A_iA_j)_S(B_jB_i)_S \\ &\quad + (A_iA_jA_k)_S(B_kB_jB_i)_S + (A_kA_jA_i)_S(B_iB_jB_k)_S]. \qquad (11) \end{aligned}$$

In every case all subscripts run from 1 to h, *and terms where some or all of the subscripts are equal do not in general vanish.* For a matrix term AxB is not in general equivalent to a *single* dyadic term $\mathbf{MN{:}X}$.

The general rule for any m is now easy to see. The quantity $p!m_p$ may be written as a *summation of square bracket expressions, each square bracket enclosing p! terms. The leading term is*

$$\Sigma A_{iS}B_{iS}A_{jS}B_{jS}A_{kS}B_{kS}\ .\ .\ .\ A_{rS}B_{rS}$$

where the p subscripts i, j, k, . . . r are those of the given matrix coefficients in (2) and may be either alike or different. The other terms are formed by first keeping the antecedents A fixed and making interchanges among the consequents B. A multiple interchange such that q consequents $B_aB_bB_c$ . . $B_eB_fB_g$ *(whether alike or not) change places among themselves leads to a product of two factors as* $(A_cA_bA_e\ .\ .\ .\ A_dA_fA_a)_S\ (B_aB_fB_d\ .\ .\ .\ B_eB_bB_c)_S$ *where the order of consequents is the order they stand after the interchange, while the order of antecedents is the reverse of that of the consequents. Any antecedents or consequents which have not been moved yield the same scalars as in the leading term.*

These quantities m may be transformed in many ways.

6. *A Second Method of Solution.*—Another method of solving the equation (2) follows at once from the analogy between θ and an ordinary dyadic. If $\varphi = \Sigma\mathbf{NM}$ let $\varphi' = \Sigma\mathbf{NM}$ corresponding to θ', that is to $\Sigma A'xB'$. *Let* $C_1, C_2, \ldots C_{n-1}$ *be n-1 matrices or dyadics such that each is orthogonal to the given matrix C in the sense that* $C{:}C_i = 0$ *for* $i = 1, 2, \ldots, n-1$. *Form* $n-1$ *new matrices* $\theta'(C_i)$. *A matrix orthogonal to all these new matrices will in general be a solution of (2) aside from a scalar factor which may be found by direct substitution.* This scalar may also be found as a function of m_n and the matrices $C, C_1, C_2, \ldots C_{n-1}$.

This method is analogous with Hamilton's original method of solving the linear vector equation.[6]

7. *Conclusion.*—The analogy between a dyadic and a vector by virtue of the double dot product may be extended to polyadics and K-tuple dot

products, hence to dyadics whose antecedents and consequents are polyadics, say double polyadics. The analogy may also be applied to certain other products. Thus the scalar $((\mathbf{ab},\ \mathbf{cd}))_S$ or $\mathbf{a.bc.d} - \mathbf{a.dc.b}$ may be regarded as a product of two dyadics $\mathbf{M}*\mathbf{N}$ and dyadics found which are orthogonal with respect to star product, that is such that $\mathbf{M}_i*\mathbf{N}_k$ is unity when subscripts are equal, otherwise zero. We may similarly have star products for double dyadics and double polyadics. We shall have Hamilton-Cayley equations with respect to star products, where the scalars m are formed by star product just as those of this paper were formed by double dot product. The present paper is intended as introduction to more detailed treatment of some of these phases of multiple algebra. They have a bearing on ordinary algebra by virtue of the fact that a polynomial of degree K in N variables corresponds to a symmetrical polyadic of order K in space of N dimensions.[7]

[1] In a series of *Paris* papers, *C. R. Acad. Sci.*, **98,** and **99** (1884) and elsewhere. The complete bibliography may be found in *Bull. Inter. Assoc. Promoting Study Quaternions and Allied Branches,* particularly for March, 1908.

[2] *Paris, C. R. Acad of Sci.*, **99,** p. 435.

[3] "Note on the Linear Matrix Equation," *Edinburgh Math. Soc. Proc.*, **22,** 1904 (49–53).

[4] The laws of signant and non-signant indices for p-way determinants were given by Lepine Hall Rice, *Amer. J. Math.*, **40,** No. 3, July, 1918.

[5] By forming the determinant of $\varphi - \lambda I$.

[6] *Elements of Quaternions*, Vol. I, 2nd. Ed., p. 498.

[7] Binary quadrics as vectors in 3-space, with what I here call star orthogonality, have been studied in great detail by Emil Waelsch. See the bibliographies referred to in Note 1. His method is given in *Wien. Ber.*, **112,** 1903, p. 645, 1091, and 1533, and called "Binäre-analyse."

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NOTE ON X-RAY SPECTRA

By William Duane and R. A. Patterson

Jefferson Physical Laboratory, Harvard University, and Sloane Physical Laboratory, Yale University

Communicated March 20, 1922

In previous papers[1] the authors have presented measurements of the critical absorption wave-lengths belonging to the *L* series of nine of the heavy elements (27 wave-lengths in all). We have also discussed the bearing of these data on theoretical matters, and their relation to other available measurements of X-ray spectra. Since these papers were published, many accurate measurements have been made of the emission spectra in both the *L* and *M* series for the heavy elements.

Coster,[2] working in Siegbahn's laboratory, has accurately measured the *L* series emission lines for eleven heavy elements. He has extended the three known *M* absorption limits to five for uranium and thorium, and has measured the first three *M* limits for bismuth.

De Broglie and Dauvillier[3] have studied in detail the *L* series emission spectra for six heavy elements.

With these new, accurate data, Smekal,[4] Coster,[2] Dauvillier[3] and Wentzel[5] have constructed atomic systems to explain the source of each known emission line. The systems are based on the Bohr-Sommerfeld theory of the origin of characteristic X-radiation. In developing and discussing their systems, these investigators have used the values of the *L* absorption limits measured by us. Certain criticisms have been made of a few of our values. It has therefore seemed desirable to measure some of the limits again.

Coster[6] has suggested a possible source of error in our values for the absorption limits of tungsten on the ground that we used an X-ray tube equipped with a tungsten anticathode, and that therefore we would get an effect not due to absorption alone but also to the neighboring emission lines. We wish to call attention to the fact (stated on page 534 of our paper in the *Physical Review*, Dec. 1920) that we took[7] the precaution of using an X-ray tube with a molybdenum target to avoid exactly this difficulty. If there had been an appreciable amount of tungsten on the anticathode due to sputtering, we would have detected the presence of the strong emission lines of the L series. In a more recent paper Coster[8] states that he has measured the L_1 and L_2 absorption limits of tungsten and obtained essentially the same wave-lengths as those given by us.

Table I contains our new values for the L_2 and L_3 absorption limits of Pt, Au and Bi. These measurements were taken with almost the same arrangement of apparatus as before. The X-ray tube was provided with an arm which reached nearly to the first slit of the spectrometer, thus reducing the absorption of X-rays by air. At the end of this arm and also on the front of the ionization chamber thin mica windows were employed which still further reduced the absorption of X-radiation. Finally the sensitivity of our electrometer had been increased many times. We were thus supplied with a much more sensitive detector of X-ray spectra than in our earlier measurements. This enabled us to use narrower slits and to obtain at the same time greater drops in our ionization currents for the absorption limits.

TABLE 1

CRITICAL ABSORPTION. L SERIES OF X-RAYS

GRATING SPACE FOR CALCITE, $2d = (6.056 \pm 0.004) \times 10^{-8}$ CM.

CHEMICAL ELEMENT	L_2		L_3	
	$\lambda \times 10^8$ cm.	ν/ν_∞	$\lambda \times 10^8$ cm.	ν/ν_∞
Pt	$.9318 \pm 3$	0.9779	$.8918 \pm 9$	1.022
Au	$.9008 \pm 4$	1.0116	$.8610 \pm 9$	1.058
Bi	$.7871 \pm 4$	1.1578	$.7562 \pm 9$	1.205

In the first column for each limit is given the wave-length in Ångstrom units; in the second, the wave number divided by the Rydberg constant.

Our new value for L_2 of Au differs from our previous one by 0.17 per cent. The error in our previous work was due to the proximity of the bromine ionization limit, which has a wave-length $= 0.91796 \times 10^{-8}$ cm. In our new measurements it was unnecessary to have ethyl bromide in the ionization chamber.

For the L_3 limits to Pt and Bi the new values differ from the old ones by a considerable amount in excess of our then estimated error. These errors were undoubtedly inherent in the faintness of the effect which we were endeavoring to measure.

The other three values are substantially the same as we gave before. We do not find the error in L_3 for Au suggested by Dauvillier.

At the time of our previous investigation, the available evidence seemed to indicate that the β_5 and γ_2 emission lines belonged, respectively, to the L_1 and L_2 groups—i.e., that β_5 was emitted when an electron fell into the L_1 orbit, and γ_2 when an electron fell into the L_2 orbit. As we definitely proved that the L_1 and L_2 absorption limits for tungsten were of lower frequency than β_5 and γ_2 (a fact since verified by Coster), the idea of absorption limits being "limiting frequencies" for groups of emission lines was confronted with an anomaly. We proposed an explanation of this apparent anomaly that was in accord with the Bohr theory. Since then, however, Hoyt[9] and Dauvillier[3] have pointed out that the W emission line at $\lambda 1.212$ should be taken as β_5 instead of the one at $\lambda 1.203$ as before. This suggestion has been confirmed by the more accurate measurements of the corresponding line for the other heavy elements. When the square root of the frequency is plotted against atomic number for the β_5 lines, the points all lie on a smooth curve. Furthermore, in place of γ_2, the work of Dauvillier and Coster indicates that a new γ line, γ_6, at $\lambda = 1.072$, is the result of an electron falling into the L_2 orbit, and therefore belongs to the L_2 group. Both the old β_5 line for W and the γ_2 lines have been assigned to the L_3 group.

Consequently, we now take β_5, γ_6 and γ_4 to be the shortest emission wave-lengths in the L_1, L_2 and L_3 groups, respectively.

Coster has compared his new values for the wave-lengths of each of these emission lines with our values of the corresponding absorption limits, and has concluded that the absorption limits are in general somewhat shorter in wave-length than the emission lines associated with them. Table II gives this comparison using our new values for L_2 and L_3.

TABLE II

ELEMENT	L_2	γ_6	L_3	γ_4
Pt	.9318 ± 3	.9317	.8918 ± 9	.8950
Au	.9008 ± 4	.90125	.8610 ± 9	.8663
Bi	.7871 ± 4	.7874	.7562 ± 9	.7610

Within experimental error the wave-lengths of the absorption limits are shorter than the wave-lengths of the shortest emission lines associated with them.

Coster[2] now places the number of L doublets at four, $\eta - l$, $\beta_2 - \alpha_2$, $\gamma_5 - \beta_6$, $\gamma_6 - \beta_5$. The $\gamma_1 - \beta_2$ doublet has been eliminated which is in accord with our suggestion that the electrons responsible for these two lines do not appear to start from the same orbit but different ones. He has also found and measured the two M absorption limits predicted by us for U and Th in order to explain the existence of the $\eta - l$ doublet. His values are in close agreement with our predicted ones.

According to Smekal,[4] the $M\gamma$ line is due to the passage of an electron from an N_3 orbit to an M_3 orbit. If this is true, the following relation between frequencies should hold:

$$(L_3 - L_1) - M\gamma = L\beta_3 - L\beta_2$$

Coster has tested this relation using his new values for β_2 and β_3, Stenstrom's value for $M\gamma$ and our values of L_1 and L_3. He finds it confirmed by the available experimental data except in the cases of Pt and Bi. Our new values bring Pt into line. For Bi, however, our correction is in the right sense but far too large. We cannot at present explain this discrepancy. Table III gives our new values for this relation.

TABLE III

ELEMENT	$L_2 - L_1$	$M\gamma$	$(L_3 - L_1) - M$	$\beta_3 - \beta$
Pt	.1706	.17158	−.0010	.0000
Bi	.2163	.20144	+.0149	+.0176

The lines β_7 and γ_4 have been assigned by Coster to electrons which start from *different* outer orbits and fall, respectively, into L_1 and L_3 orbits. Assuming the γ_4 electron to start from a higher orbit than β_7, we should then expect $\gamma_4 - (L_3 - L_1)$ to be greater than β_7. Coster shows this to be the case except for Pt and Bi. Our new measurements bring both Pt and Bi into line with the other elements. Table IV contains values for these quantities in the cases of Pt and Bi.

TABLE IV

ELEMENT	$\lambda_4 - (L_3 - L_1)$	β_7	Δ
Pt	847.8	844.9	+2.9
Bi	980.6	(977.2)	+3.4

The value of β_7 for Bi was obtained, as suggested by Coster, by interpolation based upon the constant difference in wave-length exhibited by the elements for $\beta_2 - \beta_7$, the wave-length for β_2 being known. The differences in the fourth column are in the same sense and of the same magnitude as those found by Coster for Ta, W, Au, Tl and Pb. Moreover, if we estimate a value for the wave-length of γ_4 for U based upon our value of L_3 for the same element, we obtain a difference greater than those above and also in the same sense. It is therefore quite well established that the β_7 and γ_4 electrons start from orbits which maintain approximately the same difference in energy level as we go from Ta to U.

The authors in 1920 compared the frequencies of certain L emission lines with the differences in frequency between corresponding L and M absorption limits.[1] Kossel first pointed out that these quantities should be equal on Bohr's theory. At that time we found each L emission frequency to be equal to a corresponding difference in absorption frequencies within the limits of experimental error. Coster[2,8] has recently extended this comparison with his new and more accurate measurements to in-

TABLE V

ELEMENT	l	$L_1 - M_5$	η	$L_2 - M_5$	α_1	$L_1 - M_1$	α_2	$L_1 - M_2$
U	855.84	855.1	1134.95	1131.8	1003.23	1002.17	990.37	989.21
Th	819.19	818.8	——	——	955.78	954.80	944.08	943.15
Bi	——	——	——	——	798.54	797.43	790.20	789.35

	β_1	$L_2 - M_2$	β_3	$L_3 - N_3$	β_4	$L_3 - M_4$
U	1269.08	1265.87	1286.29	1285.78	1222.53	1220.86
Th	1194.94	1193.15	1211.67	1209.77	1155.00	1151.86
Bi	959.93	958.16	973.85	971.2	——	——

clude two more M absorption frequencies for U and Th, and the first three M limits for Bi. Table V contains his values with the exception of $L_3 - M_3$ for Bi which we have corrected to accord with our new value of L_3 for that element.

In each case the emission frequency exceeds the difference in absorption frequencies, but by an amount that is less than 0.3 of a per cent.

The K Series of Molybdenum.—The authors have also measured the emission and absorption spectra for the K series of molybdenum. The apparatus and method used are described in previous papers.[7] A Coolidge tube equipped with a molybdenum target and an extra arm at the end of which happened to be a thin glass window served as the source of radiation.

Table VI contains the results of these measurements.

TABLE VI

K Series of Molybdenum

Grating Space of Calcite = $2d$ = 6.056 $(10)^{-8}$ Cm.

LIME	NO.	GLANCING ANGLE	WAVE-LENGTH × $(10)^8$ CM.	ORDER
α_2	3	13°–36′–08″	0.71213 ± 8	2d
α_1	3	13°–31′–07″	0.70783 ± 7	2d
β'	3	12°–02′–50″	0.6320 ± 2	2d
β	6	12°–01′–50″	0.63114 ± 7	2d
$\beta\beta'$	1	5°–59′–05″	0.6314 ± 3	1st
γ	3	11°–48′–32″	0.6197 ± 2	2d
γ	1	5°–52′–30″	0.6199 ± 3	1st
K	1	5°–51′–40″	0.6184 ± 2	1st

In the second column the number is the number of actual measurements made.

In the second order, the β line was markedly asymmetrical and we were able to estimate the position of its weaker component, β', predicted by Sommerfeld. The wave-length difference between these two lines is of the same order of magnitude as that found in the case of other elements. The wave-lengths of β and γ found in the first order are definitely longer than those found in the second order. This is chiefly due to the complex structure of both of these lines, although several investigators including the authors have found a similar but smaller effect in the case of emission lines supposed to be single.

In a recent paper Overn[10] has also given wave-lengths for these same emission lines. His values, however, do not agree with ours even when corrected for the difference between the rock salt grating space used by him and the calcite grating space used by us. The discrepancy between his values and ours seems to increase as we go farther from the "white line" used as a reference line by him.

As we determined these wave-lengths by the ionization method, we are also able to give the ratio of the intensities of α_1 and α_2. From three determinations in the second order we obtain for this ratio 1.97 as compared with 2.00 predicted by Bohr.

The ratio of the intensities of β to γ in the first order is 6.3 (one determination) and in the second order is 5.46 (two determinations).

[1] Duane and Patterson, *Proc. Nat. Acad. Sci., Washington,* Sept. 1920; *Phys. Rev.,* Dec. 1920, p. 526.

[2] Coster, *Z. Phys.,* **5,** 1921 (139); also **6,** 1921 (185).

[3] De Broglie and Dauvillier, *Paris, C. R.,* **173,** p. 137. Dauvillier, *Ibid.,* **172,** also **173,** p. 647, also other papers in this volume.

[4] Smekal, *Z. Phys.,* **5,** 1921 (91, 121). Also **4,** 1921.

[5] Wentzel, *Ibid.,* **6,** 1921 (2).

[6] Coster, *Ibid.,* **5,** 1921 (147). Also *Physic Rev.,* Sept. 1921.

[7] Duane and Patterson, *Physic Rev.,* **16,** 1920 (534).

[8] Coster, *Ibid.,* Jan. 1922.

[9] Hoyt, *Proc. Nat. Acad. Sci., Washington,* Nov. 1920.

[10] Overn, *Physic. Rev.,* Nov. 1921, p. 350.

A NEW METHOD OF USING X-RAYS IN CRYSTAL ANALYSIS

By George L. Clark and William Duane

Jefferson Physical Laboratory, Harvard University

Communicated April 4, 1922

The procedure described in this paper differs in some fundamental details from the elegant methods of analyzing crystals by means of X-rays developed by Laue, the Braggs and others.

The substance to be examined may be in the form of powder or of whole crystals. In the latter case the crystal need not be large. One having a volume of only .25 cubic mm. suffices, although slightly more accurate data may be obtained, if it has a somewhat larger volume.

In our method we employ X-rays belonging to the continuous spectrum, and therefore containing rays of a great variety of wave-lengths, and not the line spectra of definite wave-lengths, as in most other methods. This procedure has certain advantages. An ordinary X-ray tube with a tungsten target may be used, and high voltages may be applied to it. More intense X-radiation can be obtained from tungsten than from such chemical elements as rhodium and molybdenum. Further in some cases it is desirable to employ X-rays of shorter wave-lengths than those in the *K* series of rhodium and molybdenum, for these shorter rays are more penetrating than the longer ones. As a rule we use the spectrum between .12 and .80 Ångström.

In making an analysis of a small crystal we place it at the center of an X-ray spectrometer, provided with an ionization chamber and a quadrant electrometer. The slits of the instrument should be fairly narrow, and the best results can be obtained by turning the X-ray tube so that the rays that pass through the instrument leave the target almost grazing its surface. This gives a narrow beam of rays. We set the ionization chamber so that the line from the axis of rotation to the chamber's slit makes an angle, 2θ, of convenient size (usually about 10°) with the direct beam of rays. Keeping the ionization chamber always in the same position we turn the crystal around by successive small steps and measure the ionization current at each step. When a set of parallel planes in the crystal lie at an angle θ to the incident beam of rays the planes pick out of the beam X-rays of certain definite wave-lengths, λ, and reflect them into the chamber, and we get a marked increase in the ionization current. Evidently the angles between the positions of the crystal that correspond to maximum ionization currents are the angles between the crystal planes.

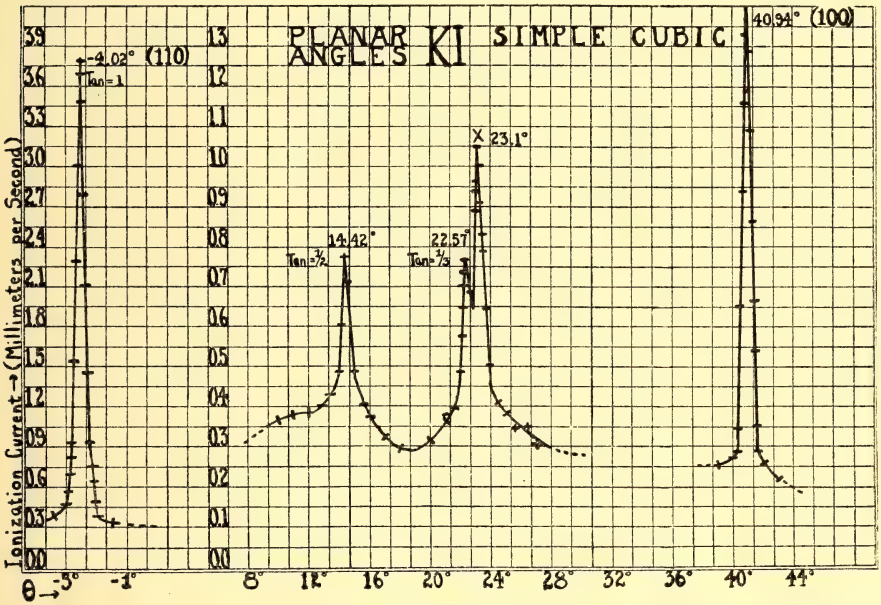

FIGURE 1

The curve in figure 1 represents the ionization current as a function of the position of the crystal in an experiment with potassium iodide. The salt came from Professor Baxter's laboratory, and was of exceptional purity. The peaks on the curve are sharply marked, and their angular positions can be estimated to within less than 0°.01. The angles between several pairs of peaks are 44°.96, 18°.37 and 26°.52, which correspond very closely with the angles between some of the principal planes in a cubic lattice, namely, 45°, 18°.43 and 26°.57. This together with a similar experiment performed with the crystal turned through 90° indicates that *KI* has a cubic lattice, which agrees with previous work by Davey and Wyckoff. The accuracy with which the experimentally determined angles agree with those given by solid geometry amounts in reality to a test of the precision with which the instrument has been set up, i.e., so that an axis of the crystal lies parallel to the axis of rotation, and so that the center of the incident beam of rays and the effective reflecting plane of the crystal lie on the axis of rotation.

One peak of the curve, marked *X*, does not correspond to any of the chief planes of the crystal, i.e., those with small indices. Its position shows that the corresponding planes make an angle of 17°.84 with the 100 planes. Possibly the marked reflection at this angle may be due to the characteristic line spectrum of iodine. Further evidence on this point will be given in a later paragraph.

The wave-lengths, λ, of the X-rays reflected at the angle θ, corresponding to any one of the peaks, are given by the fundamental law of X-ray reflection, which may be expressed by the equation

$$n\lambda = 2d \sin \theta, \qquad (1)$$

where n is a whole number, and d is the distance between successive planes. We cannot use this equation alone to determine d, for we do not yet know the value of λ. The peaks on the curve in figure 1 give us the angles between various planes only.

In order to determine the distance, d, we make use of the fact[1] that, in order to produce an X-ray of wave-length γ, the voltage, V, applied to the X-ray tube must have a value at least as large as that given by the equation

$$Ve = \frac{hc}{\lambda}, \qquad (2)$$

where h is Planck's radiation constant, e the charge of an electron (e) and c the velocity of light. This equation may be written in the form

$$V\lambda = \frac{hc}{e} = 12{,}354, \qquad (2')$$

where the numerical value[2] of the universal constant, hc/e, corresponds to V expressed in volts and γ in Ångströms.

In an actual experiment to measure d we set the crystal so that the X-rays reflected by the planes we desire to investigate enter the ionization chamber (i.e., we set on one of the peaks in the curve of figure 1), and reduce the voltage until the ionization current sensibly vanishes. The voltage at which it vanishes is sharply marked, especially if the voltage is a constant one, from a storage battery, for instance. It is a good plan to increase the current through the tube at the same time as the voltage decreases in such a way as to keep the product of the two, that is, the power delivered to the tube, constant. Evidently the ionization current will not completely vanish until the voltage becomes so low that it cannot produce the *longest* X-rays reflected at the angle θ according to equation (1). The longest waves correspond to $n = 1$.

A small correction has to be added to the angle θ due to the fact that the source of rays and the slit of the spectrometer have finite breadths. The rays in the incident beam are not all exactly parallel to each other and the correction equals half the angle between the rays that deviate most from the central ray on its two sides. This angle is the breadth of the source plus the breadth of the slit divided by the distance between them. The correction is usually small. In our experiments it amounts to about 3′ of arc, which is less than one per cent of the corrected value of $\theta = 5°.51$ used in the first illustrative case described below.

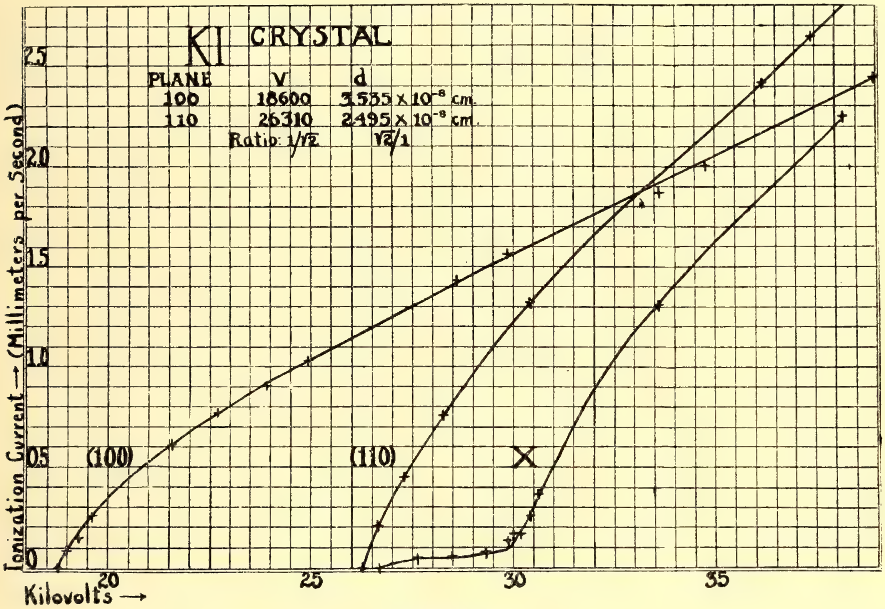

FIGURE 2

The curves in figure 2 represent the ionization currents as functions of the voltages for three of the peaks of figure 1. The voltages at which the ionization currents vanish for the peaks marked 100 and 110, respectively, are 18,600 and 26,310 volts.

Eliminating λ between equations (1) and (2′) we get, with $n = 1$,

$$d = \frac{12,354}{2V\sin\theta}. \tag{3}$$

Substituting in this $\theta = 5°.51$ and the values of V, we find for d the 3.535 and 2.495 Ångströms, respectively. The ratio of the first to the second is 1.417, which differs little from $\sqrt{2} = 1.414$, the value it should have in a cubic crystal. Taking 3.532 as the average length of the edge of a cube, and assuming one atom to each cube the calculated density of KI comes out 3.111, in close agreement with the density 3.114 measured by Baxter.

Equation (3) contains a universal constant $\frac{hc}{e} = 12{,}354$ and two quantities, a voltage, V, and an angle, θ, both of which we measure. The values of d, therefore, determined by this method are independent of previous measurements of the distance between the planes in any crystal. This advantage is, perhaps, more theoretical than practical at present, for measurements with X-rays reflected from calcite probably furnish us the most accurate estimates we have for the universal constant.

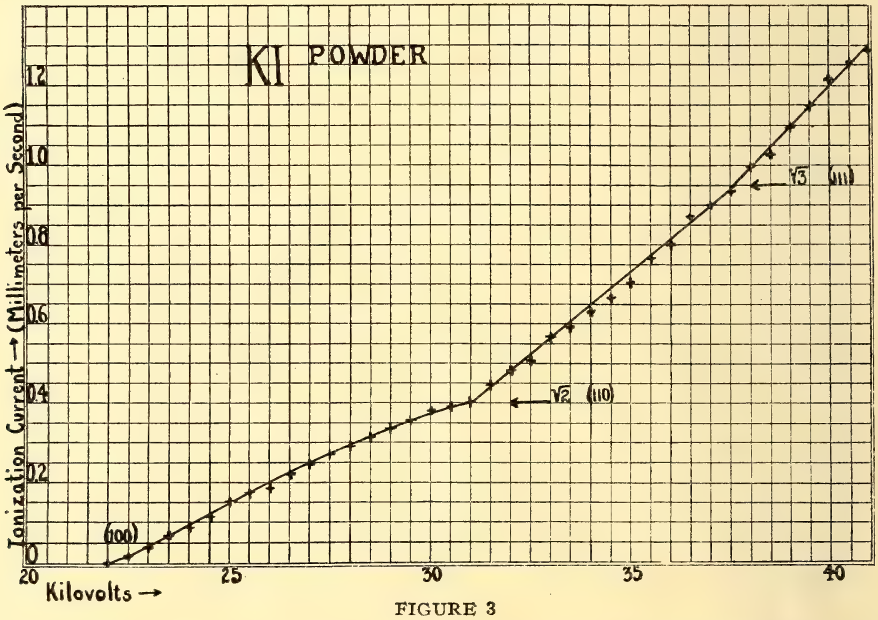

FIGURE 3

The agreement between the distances, d, determined by this method with those calculated from the density, atomic weights, etc., helps to confirm the general correctness of the laws of X-rays as at present understood.

If the specimen to be examined has the form of powder, we place the powder at the center of the spectrometer, fix the position of the ionization chamber at an angle 2θ and reduce the voltage by steps making measurements of the ionization current as before. The ionization current vanishes when the voltage is no longer able to produce the longest X-rays that are reflected at the angle θ by the powder. This maximum wave-length corresponds to $n = 1$ in equation (1) and to the greatest distance d between successive planes in the crystal that are alike. The curve in figure 3 represents the ionization current as a function of the voltage in an experiment with potassium iodide powder. In addition to the point where the ionization current vanishes there are several well-marked places where the curvature changes abruptly. The voltages at these breaks in the curve are the voltages at which the rays reflected from other crystal planes cease to be produced. These voltages together with the angle θ, substituted in equation (3), give the corresponding distances between the planes. The three most sharply determined voltages are 21,740, 31,010 and 37,600 volts, respectively. The ratios of these numbers to each other differ from $\sqrt{2}$ and $\sqrt{3}$ by small fractions of one per cent only. They correspond, therefore, to the 100, 110 and 111 planes, respectively, of a cubic lattice. The angle θ equals $4°.51$ in this experiment, and the average value of d, calculated from equation (3), is 3.534, in close agreement with that obtained in the above-described measurement with a small crystal of potassium iodide.

The curve marked X in figure 2 corresponds to the peak marked X in figure 1. There appears to be a sharply indicated point of inflection in the former at a voltage of 30,000 volts. Substituting this in equation (2′) we find that the corresponding wave-length is .412. This wave-length is a good mean value of the wave-lengths of α and β lines in the K series of iodine, namely, .437 and .388. 30,000 volts do not suffice to produce the K series of iodine. If, however, one sets a spectrometer on a characteristic line and reduces the voltage, the ionization current does not vanish when the voltage ceases to produce the characteristic X-rays. It vanishes only when the voltage becomes insufficient to produce an X-ray in the continuous spectrum of equal wave-length to that of the characteristic line. We have tested the existence of the peak X by turning the crystal as in the experiments represented in figure 1 with voltages in the neighborhood of 33,030 volts, which is the critical voltage for the production of the K series of iodine. We find no evidence of a peak at a voltage slightly below 33,000. The peak is just observable, however, if the voltage equals 33,500 volts. These points strongly favor the idea that the characteristic K radiation of iodine produces the peak X. The distance between the planes making an angle of $17°.84$ with the 100 planes, however, does not equal that calculated from equations (1) and (3). The former is 1.088 and the latter is 2.144, i.e., about twice the former. Perhaps this may be due to the fact that only half the atoms are iodine. If the characteristic radiation does produce the peak, and, if we adopt the interference of scattered rays as the explanation of X-ray reflection, we must assume that, on the average at least, definite time relations must exist between the passage of the primary X-ray across an atom and the emission of a secondary *characteristic* ray. If we adopt the quantum point of view, we must suppose that the direction in which the characteristic ray travels depends upon neighboring atoms as well as upon the angle of incidence of the primary beam.

The relative reflecting power of different planes may be determined by reflecting from them the *same* portion of the continuous spectrum, or different portions having known relative intensities. The use of the continuous spectrum for this purpose instead of the line spectrum has the advantage of flexibility. One can employ short, penetrating X-rays, and with these investigate dense crystals of high absorbing power.

[1] *Phys. Rev., Ithaca,* Aug. 1915, p. 166.

[2] *J. Optical Soc.,* May 1921, p. 386.

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*ACOUSTIC TOPOGRAPHY VARYING WITH THE POSITION OF THE ORGAN PIPE**

By Carl Barus

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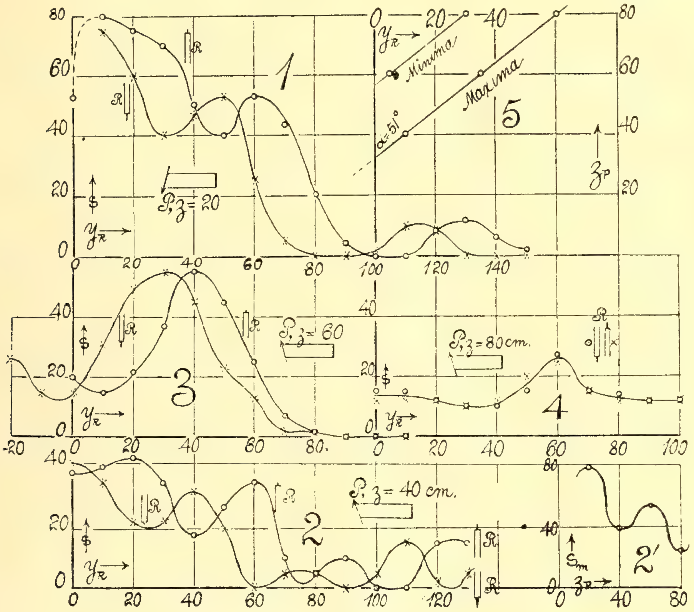
Communicated March 20, 1922

1. *Introductory.*—The coördinate axes have the x-y plane on the large table, the origin near one corner. The survey is made between walls at $y = 174$ cm. and $y = -130$ cm.; and the pipe is displaced both in the y and z (upward) direction. In the graphs, the letters P, R, T, T', used either as subscripts or directly, will refer to the pipe, the resonator, the edges of the table. Throughout s is the fringe deflection of the U-tube interferometer, proportional to the acoustic pressure or to the nodal density evoked at a given place ($x = 0, y, z$), by the sounding pipe P.

The latter was a closed brass f'' pipe, blown by the pipe blower elsewhere described, rigidly attached. In graphs 1–3, this is sketched in under P and an arrow shows the direction of the wind current. The closed cylindrical pinhole resonator R, lying with its axis parallel to y, in the yx plane (table) and moving end-on along y, is also suggestively indicated, showing the resonator azimuth for which the usually paired graphs apply. In the earlier work (figures 1–3) the *middle* of the resonator was used for the location of points in y. An inversion of the resonator thus gave the two curves for each case and they were obtained consecutively. If the mouth of R is used to determine the coördinate y as in figures 4, 6 to 10, but a single curve is obtained, for all azimuths of R. The same is true for a resonator placed with its mouth at the mean y_R of a given ordinate, s, in case of figures 1–3.

2. *Pipe Elevated above the Origin on the Table.*—In figures 1–4 the organ pipe, with its mouth vertically above the origin, was raised succes-

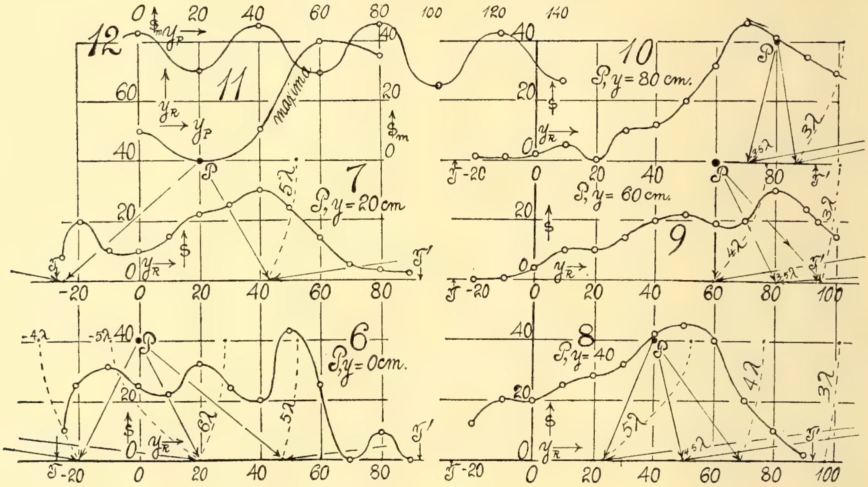
sively in steps of 20 cm. from $z = 20$ cm. to $z = 80$. For each pipe level the distribution of nodal intensity along the y axis was explored by moving R along it (with observations taken every $\Delta y_R = 10$ cm. apart). In figure 1, $z_p = 20$ cm., there is probably some distortion at the origin, owing to the wind from the near pipe, otherwise the first crest would be in negative y. From the glancing incidences, this graph has less detail than figure 2 ($z_p = 40$ cm.), which preserves the same general character but is lower in s and shrunk in the y direction. In figure 3, $z_p = 60$ cm., the topography is totally changed. There is now a trough near the origin and (on compounding) but one marked crest. Figure 4, $z_p = 80$ cm., is redeveloping a crest at the origin, but otherwise preserves the former outlines. In this figure and the following, the mouth of the resonator was used to define y_R so that the two curves are practically coincident.



If the highest crests correspond to like path differences and therefore belong together, their position y_R, z_p in figure 5 is seen to progress roughly along a straight line $y = 1.25z - 40$, the slope dy/dz corresponding to an angle of incidence of $\alpha = 51°$. The minima determine parallel lines, the vertical distance apart of consecutive lines being here $\Delta z_p = 24$ cm. or about half a wave-length.

Furthermore, the graphs, as a whole, are alternately high and low in intensity, s. If the intensity s_m of the highest crest be expressed in terms of the pipe elevation z_p, the resulting graph, figure 2′ (scale $^1/_2$), presents the same strongly marked harmonic character of the reflection graphs of the preceding paper. Cf. § 3.

3. *Pipe Displaced in a Given Level along y.*—Figures 6–10 are constructed on the same plan as the preceding, the ordinates s, denoting fringe deflections or the nodal intensity, encountered by the mouth of the resonator at y_R. If the ordinates be also taken as z_p, the pipe position, P, etc., may also be indicated on the graphs. T, T' show the edges of the table. In figure 6, where the pipe is 40 cm. above the origin, the graph is highly detailed with crests mostly 30 cm. apart. The hyperbolas for reflections from the wall at y = 174 cm. and beyond T' have been sketched in at 5λ and 6λ of path difference, together with the corresponding acoustic rays. So also those for reflection from the wall at y = − 130 cm. (beyond T). The last agrees badly with the crest observed at about −10 cm. The alluring possibility of an intersection of the positive and negative hyperbolas resulting in very high crests does not work out. In figure 7 also the positive hyperbola 5λ fairly corresponds to the observed crest, but not the negative one. In figures 8, 9, 10 the positive hyperbolas 3λ, 4λ, 5λ, also, fail of coincidence with the observed crests. These in fact lie nearly between them, or half a wave-length of path difference beyond. Thus it seems improbable that the maxima and minima can be explained as resulting from the reflection at these walls, while all other walls and the ceiling are far more distant.



In figures 6–10 one is often in doubt as to which of the crests belong together. Crests seem rather to separate and recombine. Taking the highest in the successive curves, figure 11 shows that the relation of pipe and resonator positions for maxima is here not simple as in figure 5. In fact the pipe approaches the high crest and finally overtakes it (figure 10).

There is no symmetry of graph at the middle of the table ($y = 33$) and the two edges T, T' show no correspondence. The graphs may actually be traced into free air beyond the table, as indicated in the prolongation of figure 12. Finally if the intensity s_m of the supposedly chief maxima be laid off in terms of the pipe position an harmonic figure 12 of relatively constant amplitude appears in reasonable contrast with the curve of diminishing amplitude in figure 2', for the raised pipe. In both cases, the steps of 20 cm. between pipe positions are too large for sharp discriminations; but it is noticeable that the distance between crests in figures 2', 12 and the earlier reflection figures is about $\Delta y = 40$ cm. The prevailing Δy in the single graphs 2, 6, etc., is about 30 cm. In figure 12 the points beyond $y_p = 80$ cm. were obtained in similar investigations, here omitted.

4. *Summary.*—The above work as a whole has shown that when the reflecting plate is relatively near, to the pipe and resonator, the position of maxima and minima may be predicted as a case of ordinary interference and the wave-length computed satisfactorily. The distribution of intensity among the crests and troughs remains harmonic and has not been foreseen.

If the reflecting surfaces are relatively remote the positions of crests and troughs cannot, as a rule, be found by the same method satisfactorily. In certain instances, there seemed to be an approximate fit; but as a whole the attempt was unsuccessful and the distribution of nodal intensities equally puzzling. Acoustic topography is not symmetrical to the pipe. It appeared, however, that, when the organ pipe, definitely placed, is sounded, the occurrence of nodal surfaces of a fixed position in free air is demonstrable everywhere. Distributions between walls are consistent and differ from distributions between wall and door. It is not improbable, moreover, that such surfaces are regularly grouped and may for reasonable distances be approximately parallel. If then they are intersected obliquely, for instance by the plane of the table, and if α is the mean angle between the latter and the nodal surface, the distance between crests on the table should be $\Delta y = \lambda/(2 \sin \alpha)$. Hence this intercept, Δy, may have any value depending upon the general shape of the room. If the prevailing value $\Delta y = 30$ cm. be taken, $\sin \alpha = 24/30$ or $\alpha = 53°$ roughly. This angle between surfaces is also the angle of incidence of the rays, and has been encountered more or less closely in other instances (fig. 5). The distribution of intensities would then also depend on the fixed pipe position in relation to the given room, regarded integrally as an enveloping reflector. Such an explanation is plausible and flexible, no doubt; but it has the great disadvantage that none of the results can be reproduced by computation. The conviction that some other explanation may be found is not removed and the answer probably lurks in such curves as figures 2' and 12. This is particularly so, because the pinhole resonator is practically unresponsive to antinodes and to wave trains except within ranges from the pipe which are small as compared with the dimensions of the room as a whole.

* Advance note from a Report to the Carnegie Institution of Washington, D. C.

ON STEERING AN AUTOMOBILE AROUND A CORNER

BY A. G. WEBSTER

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Read before the Academy April 25, 1921; received April 6, 1922

The question of so arranging the steering gear of an automobile that the front wheels shall skid as little as possible owing to axes not intersecting has been dealt with by A. L. Candy (*American Mathematical Monthly*, June 1920) and leads to a geometrical discussion of a high degree of complexity. The present somewhat trifling paper has no such ambitious purpose, but was suggested by a question asked on the witness-stand as to how much an automobile must turn out in order to pass another. It was necessary to make some assumption, and this led to the question of steering in general, and the principles that should govern it. We shall consider only the question of passing from a straight course to another at right angles therewith at constant velocity, and inquire how we must steer in order not to skid.

The layman ignorant of geometry and dynamics will suppose that we describe a circle, but this is impossible, not only because it is impossible to put the helm over suddenly so as to change from zero to a finite curvature, but also because the sudden change in the centrifugal force from zero to a finite quantity would produce a wholly intolerable shock. Inasmuch as the question of transition railroad curves has received so much attention, it is thought that the treatment of this question for an automobile may have some interest to engineers. We shall for the present neglect the fact that the rear wheels do not track with the forward ones, taking that up later as a correction. Thus we shall examine the curve described by a machine of zero wheel-base, like a wheelbarrow, measuring wheel or monocycle. The condition is evidently that the curvature shall begin at zero, increase to a maximum until the tangent has turned through an angle of 45 degrees, and then return by a symmetical process.

The centrifugal force is equal to mv^2/ρ, where m is the mass, ρ the radius of curvature, v the velocity, and if there is to be no skidding this must be less than $mg\mu$ where μ is the coefficient of friction, consequently we must have at all times

$$v^2/\rho < \mu g.$$

This does not, however, determine the shape of the curve, to obtain which some assumption must be made. If we assume that the mean value of the centrifugal force or, what is the same thing, of the curvature (since v is constant) is to be a minimum we shall get no result, except a straight line, which would be a discontinuous solution, and inadmissible. We shall therefore assume that the mean square of the curvature is a minimum, that is

$$\delta \int \kappa^2 ds = \delta \int (d\theta/ds)^2 ds = 0,$$

subject to the condition

$$\int dx = \int \cos\theta ds = \text{constant, the length of chord.}$$

But this gives

$$d^2\theta/ds^2 + \lambda \sin\theta = 0,$$

where λ is some constant. But this is the equation of the elastica or curve assumed by a bent flexible rod, ruler, or spline. In fact, in order to obtain the equation of the elastica, we have to make the potential energy of bending a minimum, and as this, at each point, is proportional to the square of the curvature, the problem is mathematically the same.

If we take the chord for the X-axis, we shall find the equation satisfied if we take the curvature proportional to y, and introducing the elliptic functions of Jacobi, with c a constant of homogeneity,

$$dx/ds = \cos\theta, \quad dy/ds = \sin\theta,$$

$$\kappa = d\theta/ds = -y/c^2, d\kappa/ds = d^2\theta/ds^2 = -1/c^2 dy/ds = -1/c^2 \sin\theta,$$

$$y = a \mathrm{cn} s/c = a \mathrm{cn} u, \; dy/ds = -a/c \mathrm{sn} u \mathrm{dn} u = -2k \mathrm{sn} u \mathrm{dn} u,$$

where

$$s/c = u, \quad a/c = 2k,$$

from which

$$dx/ds = \sqrt{1 - 4k^2 \mathrm{sn}^2 u \mathrm{dn}^2 u} = 2\mathrm{dn}^2 u - 1,$$

$$x = 2c \int_0^{s/c} \mathrm{dn}^2 u du - c \int_0^{s/c} du = c[2E(u) - u] = c[2E(k,\varphi) - F(k,\varphi)],$$

where F and E are Legendre's elliptic integrals of the first and second kind, respectively. The value of k is determined by the fact that at the point of inflexion, where $\theta = -45°$, we have $u = \overline{K}$, the complete elliptic integral, and since $\mathrm{sn}\overline{K} = 1$ and $\mathrm{dn}\overline{K} = \sqrt{1 - k_2}$ we have $2k\sqrt{1-k_2} = \sin 45°$. Putting $k = \sin\alpha$ we see that $\alpha = 22°.5$. With this value for α taking values of φ at intervals of 10° from 0° to 90°, values of x and y are calculated by means of Legendre's table and the curve drawn, which is shown on the inside of figure 1. It fits very accurately a spline made of celluloid, held fast at the vertex, and merely by contact at the point of inflexion. We find the ratio of the maximum ordinate to the whole chord or base is .2757 and that the radius of maximum curvature is the chord divided by 2.1429.

The question now arises, how nearly does the curve actually described correspond with the above description? We may take as the simplest assumption that when the driver gets ready to turn he puts the steering wheel over with a constant velocity until he is half-way around, and then turns it back with the same velocity. This is to be sure not exactly true, but as the wheel is dominated rather by friction than by inertia it will do for an approximation, and is besides about what observation shows. If the effect of this is to make the curvature of the path proportional to the time or to the distance travelled since turning (which will be shown below to be nearly the case), we have

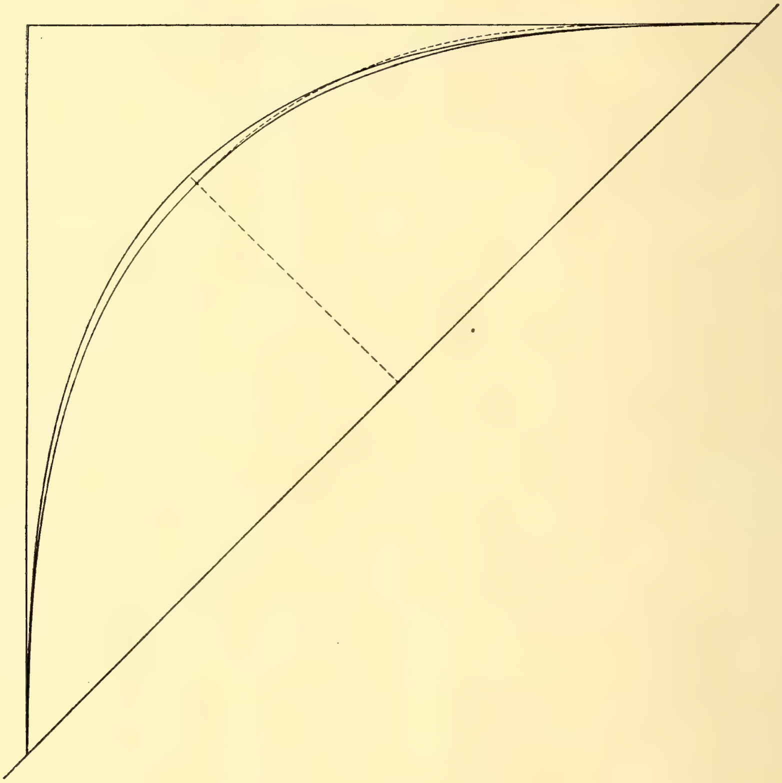

FIG. 1

$$\kappa = d\theta/ds = as$$

as the differential equation of the curve, where a is a constant. But a curve whose curvature is proportional to the length of the arc is the well-known spiral of Cornu, used in the theory of diffraction in optics. The curve may be constructed graphically from the differential equation, but more accurately by the use of tables of Fresnel's integrals. If we put

$$x = \int_e^v \cos \pi/{}_2 v^2 dv, \quad y = \int_o^v \sin \pi/{}_2 v^2 dv,$$

we have

$$dx = \cos \pi/{}_2 v^2 dv, \quad dy = \sin \pi/{}_2 v^2 dv, \quad ds^2 = dx^2 + dy^2 = dv^2,$$
$$dy/dx = \tan\theta = \tan \pi/{}_2 v^2, \quad \theta = \pi/{}_2 v^2 = \pi/{}_2 s^2 = as^2/2.$$

The integrals above have been tabulated by Gilbert, and are reproduced in Verdet's *Traité d'Optique Physique.* From them the spiral has been plotted, and is shown in the outer curve in figure 1. It will be seen that it differs very little from the elastica. As a matter of fact the ratio of the maximum ordinate at right angles to the chord is .2861 compared with .2757 for the elastica. For the circle this ratio would be only .2071. The reason for the near agreement of the two curves, in one of which the curvature is proportional to the arc, and in the other to the ordinate, is of course the fact that for a considerable distance the curve is nearly straight.

The simplest assumption that could have been made for a rough calculation would have been that of a parabola, $y = ax^2$, but the ordinary parabola of order two would not do since its curvature does not vanish at the origin, and diminishes as we leave the origin. A cubical parabola would be the most obvious, but we shall do better by putting $y = ax^n$ and determining the exponent n so that the curve shall pass through the vertex of the elastica and be tangent to it. We find $n = 3.458$. The curve is drawn dotted on figure 1, and lies between the other two curves. The ratio of the extreme distance from the chord to the length of the chord in the case $n = 3$ would have been .25. We consequently see that any of the four curves, the elastica, spiral of Cornu, parabola of order 3.458, or cubical parabola, will give a very good approximation to the actual path.

We have now to consider the effect of the finite length of the wheel-base, and this leads us to consider the question of steering in general. We shall first suppose for generality that both the front and rear wheels can be steered, as in a hook and ladder truck, but we shall still neglect the effect of the finite width of the track. The case contemplated is exactly realized in the bicycle. In figure 2, let l be the length of the wheel-base, let s, θ, α, ρ be the distance run, the angle made by the tangent with a fixed direction, the angle through which the wheel has been turned (that is, the angle made by the tangent to the path with the chassis, and the radius of curvature of the path, respectively, for the front wheel, and let the same letters with accents denote the corresponding quantities for the rear wheel. (When we come to consider the dynamics of the matter we shall need corresponding quantities for the center of mass, and shall denote them by two accents.) Then since the whole machine turns about the instantaneous center which is the intersection of the axes of the two wheels, we see on reference to figure 2, ψ being the angle made by the chassis with the fixed direction,

$$\rho d\theta = ds,\ \ \rho' d\theta' = ds',\ \ \theta + \alpha = \theta' + \alpha' = \psi.$$

We also have

$$\cos\alpha/\rho = \cos\alpha'/\rho' = \sin(\alpha - \alpha')/l,$$

which equations give ρ and ρ' as functions of α, α'. If then α and α' are given as functions, say of s and s', this makes seven equations between

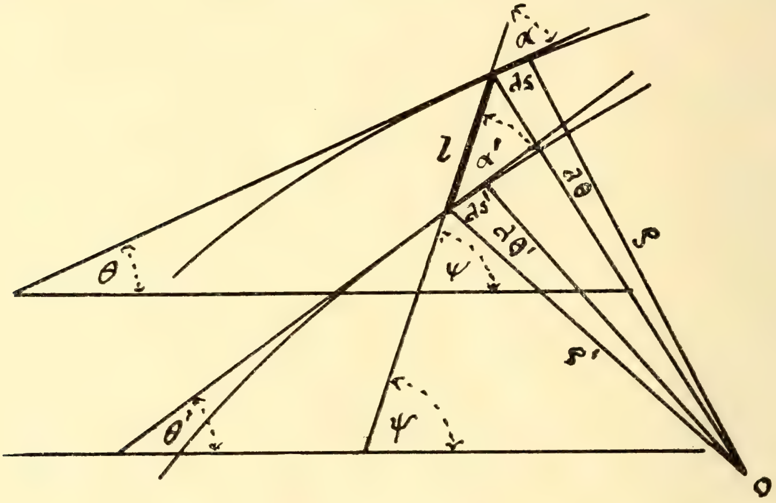

FIG. 2

the variables α, α', θ, θ', s, s', ρ, ρ', so that we may find the differential equations of the path of either the front or rear wheel. In particular, if α and α' are constant and equal, we shall have merely a movement of translation, while if $\alpha' = 0$, $\alpha =$ constant we shall describe two concentric circles. If $\alpha' = 0$, $\alpha = 90°$, the machine will describe a circle about the rear wheel. If, with these values of α, α' the length l be made variable, and a given function of s, the front wheel will describe a curve and the rear wheel its evolute. Thus a curve may be mechanically constructed from its intrinsic equation. This principle is used in certain integrating machines, and the author has invented an integrator for drawing trajectories involving this principle, which will be described later if he succeeds in having it built.

In the case of the automobile we have

$$\alpha' = 0, \qquad \rho = l\,\mathrm{ctn}\,\alpha, \qquad ds = l\,\mathrm{ctn}\,\alpha\, d\theta,$$

and if as before we assume the mode of steering to be $\alpha = as$ we may integrate and obtain the intrinsic equation of the curve,

$$l d\theta = \tan(as) ds, \qquad la\theta = \log \sec as.$$

Developing the tangent in series we may see how much this differs from the Cornu spiral. The curve has been drawn graphically from the differential equation, but since on a sixty-foot chord it differs but slightly from the spiral (which may be judged from the fact that for $\theta = 45°$, $\alpha = 23°.2'$; $\rho = 22.35$ compared with $\rho = 22.28$ for the spiral, and that the rear wheel's track is less than two feet distant at the maximum) it is not here reproduced.

It may be said that the above is not applicable to the way in which an automobile actually skids, when it is well known that the two wheels do not break the frictional constraint together, but the rear wheel generally skids first. This may be true, but I think this will not invalidate anything that I have said. One would suppose that this question would be treated in any elementary book on statics, but as I have failed to find it in any treatise, I insert a treatment here. It will now be necessary to consider a third path, namely, that of the center of gravity of the machine, which we shall still assume to be on the line joining the wheels, at distances d and d' therefrom, figure 3. Then we have $d'\text{ctn}\alpha'' = (d + d')\text{ctn}\alpha$. The three paths have the same center of curvature. The constraints due to the friction can be only at right angles to the paths of the wheels, and we will call them F, F'. The kinetic reaction of the machine is described by the centrifugal force C, the tangential reaction R, and the moment S. It may be asked how the wheels can cause any reaction in the path, but it will be seen that there must be such when we consider that energy of translation being converted into energy of rotation and conversely will give rise to acceleration in the path, negative or positive, respectively. In fact we see at once that S must vanish with R, for three concurrent

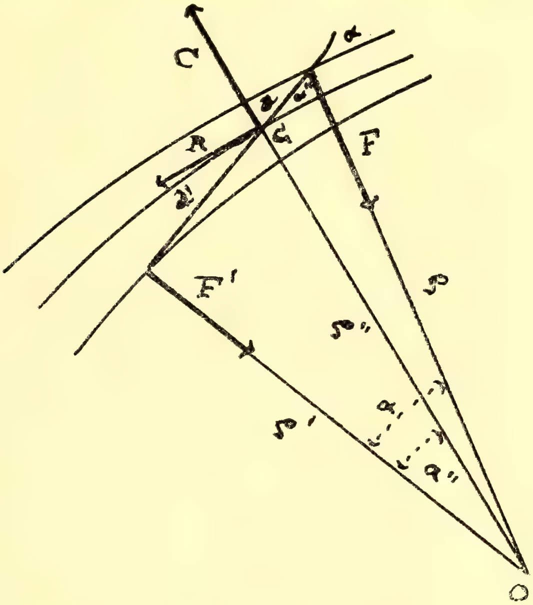

FIG. 3

forces do not give rise to any moment. Resolving parallel and perpendicular to the chassis,

$$F' + F\cos\alpha = C\cos\alpha'' + R\sin\alpha'',$$
$$F\sin\alpha = C\sin\alpha'' - R\cos\alpha'',$$

and taking moments about the center of mass,

$$F'd' - Fd\cos\alpha = S,$$

or making use of the relation between α and α'',

$$F' - F\sin(\alpha-\alpha'')/\sin\alpha'' = S/d'.$$

Solving these equations we obtain

$$R = -S/d'\sin\alpha'',$$
$$F = \sin\alpha''/\sin\alpha\ (C + S\cos\alpha''/d'),$$
$$F' = \sin(\alpha - \alpha'')/\sin\alpha\ C + S/d'\ (\sin(\alpha-\alpha'')\cos\alpha''/\sin\alpha - 1).$$

Now since $C = mv^2/\rho''$ and $S = mk^2d^2\psi/dt^2 = mk^2(d^2\theta/dt^2 + d^2\alpha''/dt^2)$ and in the method of steering assumed this is $mk^2d^2\theta''/dt^2 = mk^2v^2d^2\theta/ds^2$, if v is constant, both terms are proportional to v^2. Consequently the conclusions above stated are justified. (It is to be remarked that if v is constant, the kinetic reaction above is to be replaced by the force of traction necessary to maintain the velocity constant.) The term in C is proportional to the curvature, and the term in S to the rate of increase of curvature, the latter is greatest at the beginning and end of the curve, the former at the middle.

Finally I have made some experiments to determine the coefficient of friction μ of rubber on slimy stone, with the conclusion that it is about one-fourth. Using the criterion $v^2 < \mu g\rho$ we arrive at the conclusion that the safe speed for curves as above is somewhat less than ten miles per hour, which may account for the common traffic rule of "eight miles around the corner."

THE PROBABLE ERROR OF THE VITAL INDEX OF A POPULATION[1]

By JOHN RICE MINER

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Communicated February 14, 1922

In comparing the birth-death ratios of different communities we need their probable errors to determine the significance of observed differences.

Let N = the population of a community,
B = number of births in a year,
D = number of deaths in a year.

Then the birth-death ratio or vital index is defined as $\dfrac{100\,B}{D}$.

The standard errors of sampling of births and deaths will be:

$$\sigma_B = \sqrt{\frac{B(N-B)}{N}} \qquad\qquad \sigma_D = \sqrt{\frac{D(N-D)}{N}}$$

whence $V_B = \frac{\sigma_B}{B} = \sqrt{\frac{1-B/N}{B}}$ $\qquad V_D = \frac{\sigma_D}{D} = \sqrt{\frac{1-D/N}{D}}$

and the probable error of the birth-death ratio will be[2]

$$PE_{\left\{\frac{100\,B}{D}\right\}} = 67.449\,\frac{B}{D}\sqrt{\frac{1-B/N}{B}+\frac{1-D/N}{D}} \tag{i}$$

since $r = 0$, being the correlation of errors of sampling of births and deaths. As B and D are small as compared with N we may write this approximately:

$$PE_{\left\{\frac{100\,B}{D}\right\}} = 67.449\,\frac{B}{D}\sqrt{\frac{1}{B}+\frac{1}{D}} \tag{ii}$$

This form has the advantage of not involving N, so that if the numbers of births and deaths in a community are known but not the population, the probable error of the birth-death ratio may still be found.

For Baltimore in 1917

$N = 594{,}637$
$B = 15{,}040$
$D = 11{,}355.$

Therefore by (i)

$$PE_{\left\{\frac{100\,B}{D}\right\}} = 67.449\,\frac{15040}{11355}\sqrt{\frac{1-\frac{15040}{594637}}{15040}+\frac{1-\frac{11355}{594637}}{11355}}$$

$$= 89.34\sqrt{\frac{0.9747}{15040}+\frac{0.9809}{11355}} = 89.34\,\sqrt{.00006481 + .00008638}$$

$$= 89.34 \times .01229 = 1.10.$$

By (ii)

$$PE_{\left\{\frac{100\,B}{D}\right\}} = 67.449\,\frac{15040}{11355}\sqrt{\frac{1}{15040}+\frac{1}{11355}} = 89.34\,\sqrt{.00006649 + .00008807}$$

$$= 89.34 \times .01243 = 1.11.$$

In using these formulae it should be borne in mind that the expression for the standard deviation of an index, which has been used in their derivation, is obtained on the assumption that deviations are small compared with the mean values of the variables. This will be true if B and D are of fair size, but the formulae should not be used for very small absolute numbers of births and deaths.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 52.

[2] Pearson, K. On a form of spurious correlation which may arise when indices are used in the measurement of organs. *Proc. Roy. Soc.*, **60**, 1897 (489).

THE AVERAGE CHEMICAL COMPOSITION OF IGNEOUS ROCKS

By Frank W. Clarke and Henry S. Washington

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Communicated March 22, 1922

We have recently completed computations of the average chemical composition of the igneous rocks of various countries, of the continents, and of the earth's crust. These are to be published as a *Professional Paper* of the U. S. Geological Survey, with discussions of the characters of the various averages, of the relative abundance, distribution, association, and evolution of the elements, and other matters. Inasmuch as it is probable that this paper will not appear for a considerable time, it is thought best to publish now the various average compositions arrived at, in order to put before geologists and others some interesting data as to the chemical composition of the earth.

The data used by us were 5,159 chemical analyses of igneous rocks from all over the earth, and published between 1884 and 1913, inclusive. Analyses only of "superior" quality, and of fresh unaltered rocks, were considered. These were taken from a collection of rock analyses made by one of us.[1] In computing the averages of the various countries and the continents the sum total of each constituent was divided by the total number of analyses of rocks from the area dealt with. In the case of the earth average, both in terms of oxides and of elements, the same method was adopted for the major constituents, SiO_2, Al_2O_3, Fe_2O_3, FeO, MgO, CaO, Na_2O, K_2O and H_2O (only water above 110° was considered), while the average for the minor constituents was obtained by taking the mean of the sum total of each divided by the number of analyses (which gives too low a result) and by the number of determinations (which is apt to give too high a result).

In computing the average of the whole crust the relative proportions estimated by one of us[2] some years ago were adopted. The latest estimates of the masses and compositions of the hydrosphere and atmosphere have been used. These relative masses are as follows: Lithosphere 93%; hydrosphere 7%; atmosphere 0.03%.

AVERAGE IGNEOUS ROCKS, UNITED STATES

	1	2	3	4	5	6	7	8	9	10	11	12
SiO_2	61.41	62.98	57.81	58.54	60.37	60.03	60.09	65.80	61.39	59.97	62.49	60.77
Al_2O_3	15.20	14.74	15.43	16.36	15.26	16.16	15.94	15.52	16.24	15.90	15.71	15.44
Fe_2O_3	2.47	1.86	4.60	3.74	2.53	2.91	3.41	2.17	2.84	2.11	1.97	2.77
FeO	4.21	3.69	5.98	2.36	3.07	2.44	2.86	1.60	2.85	4.13	3.01	3.46
MgO	3.01	3.70	3.52	2.54	3.77	3.26	2.34	1.52	2.74	4.38	3.67	3.15
CaO	4.46	4.69	4.64	4.11	4.80	4.49	4.85	3.44	4.81	6.11	5.67	4.82
Na_2O	3.94	3.30	3.60	5.35	3.82	4.08	3.86	3.49	4.18	3.71	3.57	3.78
K_2O	3.30	2.97	2.31	4.68	3.99	3.81	3.90	4.10	3.04	1.55	2.03	3.23
H_2O^+	0.74	0.82	0.98	1.04	1.03	1.49	1.12	1.40	0.69	1.11	1.05	1.01
TiO_2	0.91	0.89	0.73	0.76	0.58	0.64	0.88	0.55	0.72	0.66	0.45	1.01
ZrO_2	0.01	0.01	0.00	0.03	....	....	0.01	0.01	....	....	....	....
P_2O_5	0.11	0.20	0.08	0.19	0.32	0.30	0.34	0.13	0.31	0.17	0.15	0.29
MnO	0.09	0.08	0.16	0.10	0.10	0.06	0.12	0.04	0.11	0.06	0.09	0.10
BaO	0.01	0.01	0.01	0.02	0.15	0.08	0.07	0.05	0.04	0.05	0.05	0.05
Incl.	0.13	0.06	0.15	0.18	0.21	0.25	0.21	0.18	0.05	0.07	0.09	0.12
	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

1. New England and New York. 213 analyses.
2. Appalachia (New Jersey and Pennsylvania to Georgia). 188 analyses.
3. Algonkian Region (Michigan, Wisconsin, and Minnesota). 66 analyses.
4. Arkansas-Texas, including Missouri. 50 analyses.
5. Wyoming, including South Dakota and Yellowstone Park. 134 analyses.
6. Montana, including Idaho. 133 analyses.
7. Colorado. 171 analyses.
8. Utah and Nevada. 79 analyses.
9. Arizona and New Mexico. 79 analyses.
10. Washington and Oregon. 51 analyses.
11. California. 175 analyses.
12. United States. 1351 analyses, including analyses of 12 highly femic and titaniferous rocks omitted from the other columns.

AVERAGE IGNEOUS ROCKS, EUROPE

	1	2	3	4	5	6	7	8	9	10	11	12	13
SiO_2	59.84	59.10	62.39	56.89	59.83	64.04	60.59	58.83	56.93	60.69	58.98	54.22	64.73
Al_2O_3	15.12	13.80	15.16	15.54	16.08	14.24	16.43	14.44	15.86	14.67	16.28	16.30	15.92
Fe_2O_3	3.17	3.14	2.40	2.69	3.65	3.65	2.75	3.35	3.72	2.76	2.53	4.28	3.15
FeO	3.67	4.57	3.04	3.57	3.21	3.41	2.88	3.91	3.78	3.28	3.44	5.54	2.28
MgO	3.61	4.80	3.79	4.57	2.26	1.82	2.16	4.63	4.15	4.56	3.03	5.02	1.76
CaO	4.97	5.41	3.92	4.43	3.64	3.25	3.35	5.56	6.45	4.37	5.16	7.60	4.49
Na_2O	3.73	3.55	3.70	4.56	5.10	3.70	6.11	3.45	3.61	3.52	3.72	2.99	3.69
K_2O	3.40	2.68	3.23	4.47	3.52	3.82	3.96	3.04	2.96	3.56	4.67	1.92	2.56
H_2O^+	1.24	1.44	1.46	1.19	0.75	0.84	0.71	1.40	1.45	1.68	0.94	1.17	1.03
TiO_2	0.83	0.97	0.75	1.34	1.64	0.73	0.56	0.87	0.58	0.65	0.85	0.66	0.31
P_2O_5	0.23	0.22	0.12	0.34	0.12	0.25	0.27	0.29	0.27	0.19	0.22	0.07	0.03
MnO	0.08	0.14	0.01	0.07	0.10	0.15	0.19	0.04	0.14	0.03	0.04	0.14	0.03
Incl.	0.11	0.18	0.03	0.34	0.10	0.10	0.05	0.18	0.10	0.04	0.14	0.09	0.02
	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

1. Europe. 1985 analyses.
2. British Islands. 171 analyses.
3. France, including Corsica. 163 analyses.
4. Iberia (Spain and Portugal). 33 analyses.
5. Norway, including Iceland. 107 analyses.
6. Sweden, including Spitzbergen. 206 analyses.
7. Finland, including Kola. 59 analyses.
8. Germany. 474 analyses.
9. Austria-Hungary-Bohemia, excluding Tyrol. 148 analyses.
10. Switzerland, including Tyrol. 217 analyses.
11. Italy, including Sicily, Sardinia, and islands. 276 analyses.
12. Russia, including Urals and Caucasus, excluding Finland and Siberia. 98 analyses.
13. Balkania, including Greece and Archipelago. 33 analyses.

AVERAGE IGNEOUS ROCKS, AFRICA AND ASIA

	1	2	3	4	5	6	7	8
SiO_2	58.21	54.67	50.59	56.02	61.92	61.81	59.72	61.02
Al_2O_3	15.28	16.21	15.81	15.68	15.49	15.73	16.51	15.97
Fe_2O_3	3.52	3.02	4.44	3.48	3.17	2.07	3.18	2.92
FeO	3.73	4.93	5.79	4.41	2.75	4.41	2.95	3.21
MgO	3.51	4.21	5.79	4.04	2.63	2.47	3.03	2.76
CaO	5.09	6.51	7.36	5.87	4.50	5.16	5.76	5.17
Na_2O	4.83	4.36	4.27	4.60	4.10	3.11	3.78	3.69
K_2O	3.28	3.05	2.31	3.07	3.23	2.51	3.05	2.99
H_2O^+	1.26	1.13	1.47	1.25	1.23	1.48	1.11	1.24
TiO_2	0.84	1.44	1.63	1.14	0.68	0.76	0.50	0.62
P_2O_5	0.20	0.35	0.43	0.27	0.12	0.27	0.18	0.18
MnO	0.07	0.01	0.04	0.05	0.10	0.17	0.14	0.15
BaO	....	....	....	....	....	....	....	....
Incl.	0.18	0.11	0.07	0.12	0.08	0.03	0.09	0.08
	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

1. Africa, continental. 223 analyses.
2. Madagascar, including Reunion. 140 analyses.
3. South Atlantic Islands (Azores, Canaries, Madeira, Cape Verde, Ascension, St. Helena). 56 analyses.
4. Africa and islands, including Madagascar and Atlantic Islands. 419 analyses.
5. Asia, continental. 114 analyses.
6. Japan. 69 analyses.
7. Malaysia, including Philippines. 129 analyses.
8. Asia, including continent, Malaysia and Japan. 312 analyses.

AVERAGE IGNEOUS ROCKS, AUSTRALASIA, POLYNESIA, ANTARCTICA

	1	2	3	4
SiO_2	60.16	58.04	50.03	53.66
Al_2O_3	14.74	15.30	15.51	16.82
Fe_2O_3	2.60	3.83	3.88	4.01
FeO	4.41	3.99	6.23	5.34
MgO	3.76	3.95	6.62	4.67
CaO	5.03	5.39	7.99	7.59
Na_2O	3.50	3.78	4.00	3.58
K_2O	3.03	2.93	2.10	2.32
H_2O^+	1.19	1.70	1.16	0.93
TiO_2	1.01	0.71	1.96	0.87
P_2O_5	0.26	0.19	0.25	0.17
MnO	0.15	0.09	0.15	0.03
BaO	0.02	....	....	....
Incl.	0.14	0.10	0.12	0.01
	100.00	100.00	100.00	100.00

1. Australia, including Tasmania. 287 analyses.
2. New Zealand, including dependent islands. 134 analyses.
3. Polynesia (Hawaiian Islands, etc.). 72 analyses.
4. Antarctica and adjacent islands. 103 analyses.

AVERAGE COMPOSITION OF IGNEOUS ROCKS IN TERMS OF ELEMENTS

		1	2	3
1	Oxygen	46.41	47.29	47.2
2	Silicon	27.58	28.02	28.0
3	Aluminum	8.08	7.96	8.0
4	Iron	5.08	4.56	4.5
5	Calcium	3.61	3.47	3.5
6	Sodium	2.83	2.50	2.5
7	Potassium	2.58	2.47	2.5
8	Magnesium	2.09	2.29	2.5
9	Titanium	0.720	0.46	0.33
10	Phosphorus	0.157	0.13	0.22
11	Hydrogen	0.129	0.16	0.17
12	Manganese	0.124	0.078	0.075
13	Chlorine	0.096	0.063	0.04–.025
14	Barium	0.081	0.093	0.03
15	Sulphur	0.080	0.10	0.06
16	Chromium	0.068	0.034	0.01
17	Zirconium	0.052	0.017	0.02–.01
18	Carbon	0.051	0.13	0.22
19	Vanadium	0.041	0.017	0.00x
20	Strontium	0.034	0.034	0.005
21	Nickel	0.031	0.020	0.005
22	Fluorine	0.030	0.10	0.04–.025
23	Cerium, Yttrium	0.020	.....	0.00x,x
24	Copper	0.010	.....	0.000,xx
25	Lithium	0.005	0.004	0.004
26	Zinc	0.004	.....	0.000,x
27	Cobalt	0.003	.....	0.00x,x
28	Lead	0.002	.....	0.000,x
29	Boron	0.001	.....	0.00x
30	Glucinum	0.001	.....	0.00x
		100.000	100.000	100.000

1. Igneous rocks. Clarke and Washington, 1922.
2. Igneous rocks. Clarke, *Bull.* **695,** 1920, p. 28.
3. Igneous rocks. Vogt, *Zeits. prakt Geol.*, **1898,** p. 324.

AVERAGE CHEMICAL COMPOSITION OF THE IGNEOUS ROCKS OF THE EARTH

SiO_2	59.12
Al_2O_3	15.34
Fe_2O_3	3.08
FeO	3.80
MgO	3.49
CaO	5.08
Na_2O	3.84
K_2O	3.13
H_2O^+	1.15
CO_2	0.101
TiO_2	1.050
ZrO_2	0.039
P_2O_5	0.299
Cl	0.048
F	0.030
S	0.052
$(Ce,Y)_2O_3$	0.020
Cr_2O_3	0.055
V_2O_3	0.026
MnO	0.124
NiO	0.025
BaO	0.055
SrO	0.022
Li_2O	0.008
Cu	0.010
Zn	0.004
Pb	0.002
	100.000

AVERAGE IGNEOUS ROCKS, NORTHERN NORTH AMERICA

	1	2	3	4	5	6
SiO_2	60.19	57.14	63.31	56.19	52.70	58.34
Al_2O_3	15.76	15.48	16.67	16.67	16.38	15.79
Fe_2O_3	2.87	3.93	2.18	3.31	4.58	2.21
FeO	3.67	4.72	2.46	4.65	5.83	3.99
MgO	3.16	1.55	2.25	3.39	3.27	4.57
CaO	4.80	3.51	5.05	5.34	5.29	5.66
Na_2O	3.90	6.98	3.85	4.40	4.88	3.54
K_2O	3.07	3.25	2.82	3.00	2.65	3.18
H_2O^+	1.01	1.12	0.78	1.08	1.09	1.14
TiO_2	1.01	1.50	0.77	1.25	2.20	0.69
ZrO_2	0.01	0.12	....	....	....	....
P_2O_5	0.26	0.19	0.18	0.19	0.21	0.33
MnO	0.10	0.21	0.10	0.13	0.17	0.14
BaO	0.04	....	....	0.04	0.06	0.11
Incl.	0.15	0.30	0.12	0.26	0.59	0.31
	100.00	100.00	100.00	100.00	100.00	100.00

1. North America, including Greenland, Mexico, Central America, and West Indies 1709 analyses.
2. Greenland, including East and West Greenland and Ellesmere Land. 41 analyses.
3. Alaska. 24 analyses.
4. Canada, including Alaska, East Canada, and British Columbia. 188 analyses.
5. East Canada, including Ontario, Quebec, Maritime Provinces, and Newfoundland. 99 analyses.
6. British Columbia, including Alberta. 60 analyses.

AVERAGE IGNEOUS ROCKS, MIDDLE AND SOUTH AMERICA

	1	2	3	4	5	6
SiO_2	59.11	58.70	61.34	61.06	61.85	54.92
Al_2O_3	16.58	18.30	15.20	15.17	16.51	16.29
Fe_2O_3	2.88	2.85	3.03	2.42	3.03	4.05
FeO	4.11	4.20	3.30	4.07	2.28	3.53
MgO	3.07	3.36	3.47	4.65	2.29	4.33
CaO	5.73	7.08	4.88	5.91	4.55	5.28
Na_2O	4.10	3.15	4.09	3.29	4.47	5.11
K_2O	2.32	0.99	2.69	1.74	2.69	3.57
H_2O	1.04	0.79	1.05	0.73	1.11	1.55
TiO_2	0.63	0.45	0.56	0.49	0.75	0.87
P_2O_5	0.25	0.08	0.11	0.07	0.25	0.11
MnO	0.06	0.02	0.12	0.18	0.22	0.37
Incl.	0.12	0.03	0.16	0.13	....	....
	100.00	100.00	100.00	100.00	100.00	100.00

1. Mexico and Central America. 47 analyses.
2. West Indies. 82 analyses.
3. South America. 138 analyses.
4. British Guiana. 45 analyses.
5. Andes Mountains. 56 analyses.
6. Eastern Brazil (including Paraguay and eastern Argentina). 20 analyses.

RARER ELEMENTS IN IGNEOUS ROCKS

	1	2
Rubidium	.000,x	.00x,x
Arsenic	.000,x	.00x,x
Molybdenum	.000,x	.000,0x
Tin	.000,x	.00x,x
Bromine	.000,x	.00x
Caesium	.000,0x	.000,00x
Scandium	.000,0x	.000,000,000,xx
Antimony	.000,0x	.00x,x
Cadmium	.000,0x	.000,00x
Mercury	.000,0x	.000,00x

RARER ELEMENTS IN IGNEOUS ROCKS (*Continued*)

	1	2
Iodine	.000,0x	.00x,x
Bismuth	.000,00x	.000,000,x
Tungsten	.000,00x	.000,00x
Thorium	.000,00x	.000,000,x
Columbium, Tantalum	.000,00x	.000,000,x
Uranium	.000,00x	.000,00x
Silver	.000,00x	.000,000,x
Selenium	.000,00x	.000,000,x
Platinum	.000,000,x	.000,000,0x
Tellurium	.000,000,x	.000,000,0x
Gold	.000,000,x	.000,000,0x
Thallium	.000,000,0x	.000,000,00x
Indium	.000,000,00x	.000,000,00x
Gallium	.000,000,00x	.000,000,00x
Germanium	.000,000,000,x	.000,000,000,x
Radium	.000,000,000,00x	....

1. Clarke and Washington, 1922.
2. Vogt, 1898.

ELEMENTS IN THE LITHOSPHERE, HYDROSPHERE, AND ATMOSPHERE

	1	2	3	4
Oxygen	49.19	47.80	46.68	46.41
Silicon	25.71	26.65	27.60	27.58
Aluminum	7.50	7.79	8.05	8.08
Iron	4.68	4.88	5.03	5.08
Calcium	3.37	3.49	3.63	3.61
Sodium	2.61	2.72	2.72	2.83
Potassium	2.38	2.48	2.56	2.58
Magnesium	1.94	2.01	2.07	2.09
Hydrogen	0.872	0.497	0.145	0.129
Titanium	0.648	0.684	0.696	0.720
Chlorine	0.228	0.162	0.095	0.096
Phosphorus	0.142	0.150	0.152	0.157
Carbon	0.139	0.095	0.149	0.051
Manganese	0.108	0.116	0.116	0.124
Sulphur	0.093	0.086	0.100	0.080
Barium	0.075	0.078	0.079	0.081
Chromium	0.062	0.065	0.066	0.068
Zirconium	0.048	0.050	0.052	0.052
Vanadium	0.038	0.040	0.041	0.041
Strontium	0.032	0.034	0.034	0.034
Fluorine	0.030	0.030	0.030	0.030
Nickel	0.030	0.031	0.031	0.031
Nitrogen	0.030	0.016	....	....
Cerium, Yttrium	0.019	0.020	0.020	0.020
Copper	0.010	0.010	0.010	0.010
Lithium	0.005	0.005	0.005	0.005
Zinc	0.004	0.004	0.004	0.004
Cobalt	0.003	0.003	0.003	0.003
Lead	0.002	0.002	0.002	0.002
Boron	0.001	0.001	0.001	0.001
Glucinum	0.001	0.001	0.001	0.001
	100.000	100.000	100.000	100.000

1. Average composition. Ten-mile crust, hydrosphere, and atmosphere.
2. Average composition. Twenty-mile crust, hydrosphere, atmosphere.
3. Average composition. Ten-mile crust, igneous and sedimentary rocks.
4. Average composition. Ten-mile crust. Igneous rocks.

The lithosphere is assumed to be made up as follows: Igneous rocks 95.00%; shale 4.00%; sandstone 0.75%; limestone 0.25%.

In the calculations involving the atmosphere and hydrosphere the thickness of the crust is assumed to be ten (or twenty) miles.

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* Published with the permission of the Director of the U. S. Geological Survey.

[1] H. S. Washington, "Chemical Analyses of Igneous Rocks," *U. S. Geol. Survey, Prof. Paper* **99** (1917).

[2] F. W. Clarke, "The Data of Geochemistry," *U. S. Geol. Survey, Bull.* **695,** 1920 (22 and 33).

DARK NEBULAE

By Henry Norris Russell

Research Associate, Mount Wilson Observatory, Carnegie Institution of Washington

Communicated March 14, 1922

1. It is now generally believed that many of the dark markings in the Milky Way, and dark starless regions in the sky, are produced by the interposition of huge obscuring clouds between us and the remoter stars. A long list of such dark markings has been given by Barnard,[1] who has done more than any one else to point out their importance and probable nature. In some cases, as in the Pleiades, Orion and Ophiuchus, these "regions of obscuration" merge into faintly luminous nebulosity in the vicinity of certain stars, in such a way that there can be no doubt that they lie near these stars in space.

It thus appears that the obscuring masses or dark nebulae in Ophiuchus and Scorpius are at a distance of from 100 to 150 parsecs, those in Taurus at probably about the same distance, and those in Orion some 200 parsecs from us, while the dimensions of the individual clouds are themselves measured in parsecs.

The occurrence of these three great regions of obscuration within a distance which is so small compared with that of the galactic clouds indicates that such objects are probably of great cosmical temperature.

2. These dark nebulae usually appear to be quite opaque. In some cases the stars can be seen faintly through them, apparently without much change in color; but in certain instances[2] stars imbedded in dense luminous nebulosity are abnormally red.

Of the various forms in which matter may be distributed in space, by far the most efficient in producing obscuration is fine dust, since this has the greatest superficial area per unit of mass. In a cloud composed of spherical particles of radius r and density ρ, distributed at random so that the average quantity of matter per unit volume is d, the extinction of a beam of light in passing through this cloud will be e stellar magnitudes per unit of distance, where $e = 0.814\ qd/\rho r$.

The numerical factor is independent of the physical units which are employed. The factor q is introduced to take account of the complications which occur when the size of the particle becomes comparable with the wave-lengths of light.[3] For particles more than two or three wave-lengths in diameter q is sensibly equal to unity. For smaller particles it increases and is a maximum, 2.56, when the circumference of the particle is 1.12 times the wave-length. It then rapidly diminishes and becomes nearly equal to $14/3 \times (2\pi r/\lambda)^4$ for particles of less than half this diameter. The ratio q/r is a maximum, 2.42, when the circumference equals the wave-length.

For clouds of the same mean density d the opacity reaches a sharp maximum when the particles are of this size. At the same time the absorption changes from the non-selective type to the selective type, varying as λ^{-4}. For visual light the maximum opacity occurs when the radius is 0.086 ρ. A cloud of particles of this size, and of the density of rock (2.7), will exert an absorption of one magnitude if it contains only 1/86 of a milligram of matter per square centimeter of cross-section, regardless of its thickness. If the particles are of half this size, or smaller, the selective absorption is almost as complete as for a gas, but may be nearly 100 million times as great.

Obscuration of light in space, therefore, whether general or selective with respect to wave-length, will be produced mainly by dust particles a few millionths of an inch in diameter, unless such particles form a negligible proportion by weight of the obscuring cloud.

3. It is just these particles, however, which will be most influenced by the pressure of the radiation of the stars. Calculations from more accurate data confirm Schwarzchild's conclusion that for a particle of the optimum size and the density of water, the repulsive force of the sun's radiation is about ten times the gravitational attraction, and show that for stars of the same brightness, but other spectral types, the radiation pressure will be about two-thirds as great for Class M and increase for the whiter stars, till for Class B it is fully ten times as great as for solar stars.

Dwarf stars will hardly repel dust at all, but giant stars, and especially the very luminous one of Class B, will repel it very powerfully. Only the coarser particles can come near such a star—the finer ones being driven away. This selective removal, from the vicinity of bright stars, of the particles which are most efficient in cloud formation, may explain the fact that the luminous portions of these dark nebulae, though centered upon stars, do not brighten up in their immediate neighborhood as much as might have been anticipated.

The finest dust must continue to be repelled by the stars, whatever their distance. It may congregate to some degree in interstellar regions, where the repulsive forces from stars on opposite sides are nearly equal, but it can be in no true equilibrium there, and must ultimately escape to an indefinitely great distance.

4. Some force, however, operates to hold these dark clouds together, for their outlines are often sharp. This is probably the gravitational attraction of the cloud itself.

Taking a spherical cloud as an example we find that, if its mass is M times that of the sun, and its radius R parsecs, the velocity of escape at the surface is $0.092M^{1/2}R^{-1/2}$km./sec. The internal velocity of the nebular material is known only in the case of the Orion nebula, where the luminous gas shows irregular variations in radial velocity from point to point, amounting to about 5 km./sec.[5] on each side of the mean.

For a nebula 1 parsec in diameter (which may be taken as a rough representation of the small black, almost round spot about 15′ in diameter, discovered by Barnard[6] in Ophiuchus) the mass must be 60 times that of the sun, if the escape velocity is to be 1 km./sec.

If all this matter were in the form of particles of rock of the optimum size, the extinction for light passing centrally through the cloud would be 2000 magnitudes. An extinction of 10 magnitudes (quite sufficient for opacity) would be produced if the radius of the particles were 72 μ.

Though these numerical values are largely conjectural, it appears probable that the aggregate mass contained in one of these great obscuring clouds must be very considerable—probably sufficient to form hundreds of stars—and that a sensible fraction of the whole mass must be in the form of dust less than 0.1 mm. in diameter.

It can easily be shown that any dust cloud which is impervious to light must also be impervious to particles such as those of which it is composed (and to free-moving electrons as well) in the sense that such a particle could not traverse the cloud without a practical certainty of collision. These collisions may account for the existence of dust within the clouds, even if it was not a primitive constituent.

5. The transition from these dark nebulae to luminous nebulae in the vicinity of the stars appears to occur in two ways. The first is by simple reflection of the light of the stars—which appears to occur in the nebulosity surrounding the Pleiades, the star ρ Ophiuchi, and probably in many other cases. The second is by the excitation of gaseous emission, as in the Great Nebula of Orion, which is connected with one of the greatest known regions of obscuration and itself shows signs that obscuring masses lie in front of it.

Both theoretical considerations, as suggested by the writer[6] and the facts of observation collected by Hubble,[7] indicate that the luminosity of gaseous nebulae is probably due to excitation of the individual atoms by radiations of some sort (ethereal or corpuscular) emanating from neighboring stars of very high temperature. In the Orion nebula the stars of the Trapezium (θ Orionis) appear to be the source of excitation.

There is no reason to believe that the luminous gas forms the whole, or even any large part, of the matter present within the region—only that it is selectively sensitive to the incident excitation, and therefore gives out most of the light, just as the gases (carbon compounds and nitrogen) do in the coma and tail of a comet.

If the turbulent motions of the various parts of this nebula are of the same order of magnitude in the other two coördinates as in the radial direction, they must correspond to an average proper motion of 1.5 astronomical units per year, or about $0''.8$ per century (with Kapteyn's parallax of $0''.0055$). In a million years this would carry a nebulous wisp through 2°, which is more than the whole extent of the nebula.

It appears probable, therefore, that the aspect of the Orion nebula was entirely different a million years ago from what it is now, as regards its details. There is no reason, however, to suppose that the nebula was not there. We may rather imagine that wisps and clouds of dust, carrying gas with them, are slowly drifting about. Some of them pass through the field of excitation due to the radiations from the Trapezium stars, and, when in this field, the gas is set shining—faintly near its outskirts, and without excitation of the nebular lines; more strongly, and with the nebular lines, near the middle.

According to unpublished investigations by Hubble, it appears probable that the absorbing clouds in Orion, not far from the nebula, weaken the light of stars behind them by at least ten magnitudes. The exciting radiations probably penetrate to a relatively small depth into the mass and, even if they went deeper, little of the excited light could get out again. The Orion nebula, on this hypothesis, may be regarded almost as a superficial fluorescence of the gaseous portion of this vast dark cloud, in the limited region where it is stimulated by the influence of the exciting stars.

[1] Barnard, E. E., *Astrophys. J., Chicago,* **49,** 1919 (1–23).

[2] Seares, F. H., and E. P. Hubble, *Ibid.,* **52,** 1920 (8–22); *Mt. Wilson Contr.,* No. 187.

[3] Schwarzschild, K., *Sitzungsberichte der K. B. Akad. der Wiss., Math. Phys. Kl., München,* **31,** 1901 (293–338); Proudman, *Monthly Not., R. A. S., London,* **73,** 1913 (535–539).

[4] Barnard, E. E., *Astrophys. J., Chicago,* **38,** 1913 (496–501).

[5] *Publications of the Lick Observatory, Berkeley, Cal.,* **13,** 1918 (98).

[6] Russell, H. N., *The Observatory, London,* **44,** 1921 (72).

[7] Hubble, E. P., "Annual Report of the Mount Wilson Observatory," 1921; *Year Book of the Carnegie Institution of Washington,* **1921.**

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PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES

Volume 8 JUNE 15, 1922 Number 6

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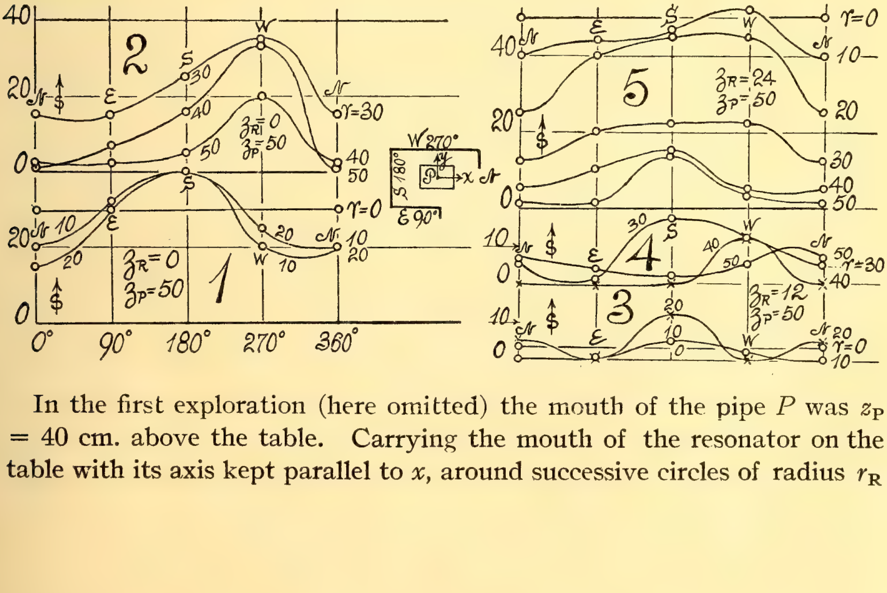
*CYLINDRICAL DISTRIBUTION OF NODAL STRENGTH AROUND THE PIPE NORMAL**

By Carl Barus

Department of Physics, Brown University

Communicated, April 14, 1922

1. *Circular and Radial Survey.*—The unexpected non-symmetric results obtained in the last paper, induced me to try a group of experiments relative to the center (nearly) of the table. The resonator was therefore carried in a circle of radius r_R around the plumb line let fall from the mouth of the pipe. The measurements were then repeated for successive levels of the pipe (z_P = 40 cm. and 50 = λ cm. nearly) and of the resonator (z_R = 0, 12 = λ/4, 24 = λ/2 cm.). The relation of the different locations is suggested by the inset of figures 1, 2, which is a reduced plan of the room, with the pipe in the vertical above P on the table. The directions + y (west) and − x (south) are thus toward walls, whereas + x (north) and − y (east) are largely toward open spaces from the table.



In the first exploration (here omitted) the mouth of the pipe P was z_P = 40 cm. above the table. Carrying the mouth of the resonator on the table with its axis kept parallel to x, around successive circles of radius r_R

= 0, 10, 20, 30, 40, 50 cm. gave intensity distributions, *s*, very definite and inter-related, but quite unsymmetrical to the pipe vertical. The *s*-oscillations in eastern and northern *R*-displacements from the center, (i. e. to open locations) were meager, while the *s*-oscillations for southern and western directions, trending toward closed parts of the room, were very marked. As *r* increased from 0 to 10, 20, 30, the maximum intensity lay toward the south; beyond this ($r = 40, 50$ cm.), it moved to the west. All the maxima of intensity were larger than the intensity on the table, under the pipe ($r = 0$); even at $r = 50$ cm. (meaning a pipe resonator distance of 64 cm.), the *s*-value was a little in excess of $s = 23$ for the normal case.

Another view of the results was given by collecting all the points on the line S-N along *x*, and on the line W-E along *y*. The features of the latter were somewhat vague, but the S-N line presented conspicuous crests and troughs. These were from 40 to 45 cm. apart, and situated on either side of, but without other appreciable reference to the pipe vertical *P* in their contours. The inference would be that we must look to other causes for their occurrence.

The mouth of the pipe was now raised in the same vertical to $z_P = 50$ cm. above the table, or a little over a wave-length (48 cm.). The data so obtained are constructed in the way described in figures 1, 2, for the rotation of the resonator and figure 6, for its corresponding $\pm x$ and $\pm y$ displacement. The central intensity ($r = 0$) has increased, because of the λ-elevation conformably with the earlier results. The curves as a whole are simpler in outline; the unique maximum is again transferred from *S* to *W* as *r* increases, this time a little sooner, in fact; i.e., before *r* reaches 30 cm. For large values of *r*, intermediate azimuths would have been desirable; but as the phenomena are adequately given by the linear surveys of figure 6, the extra labor was thought superfluous. The S-N lines in both surveys are very much alike, the maxima and minima in the case of figure 6 having been shifted into the north or positive *x*, for the elevated pipe. This also recalls certain earlier results. The distances apart of crests and troughs is here somewhat more difficult to define; but it is still much above 40 cm. and the pipe vertical is ignored as before.

For the fixed position of the pipe, it was now desirable to raise the plane of the resonator and the graphs figures 3, 4, and 5 contain the nodal intensities for an elevation $z_R = 12$ cm. (a quarter wave-length) above the table. This is the level of an antinode at the pipe normal and the curves are therefore low throughout, as in the earlier work for normal positions. Nevertheless as we depart from the pipe normal or center, to circumferences of radii 10, 20, 30 cm., the unique maxima in the south become prominent and beyond $r = 30$, they again shift to the west.

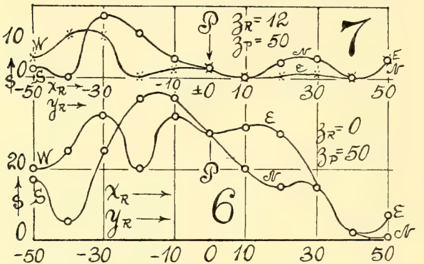
In figure 7, the S-N line has the same characteristics as in figure 6 with its two marked crests. Somewhat less certainly the resemblance in W-E lines may be detected. As in all the N-S lines the central trough lies for some reason to the right of the pipe normal. The maxima are far apart apparently.

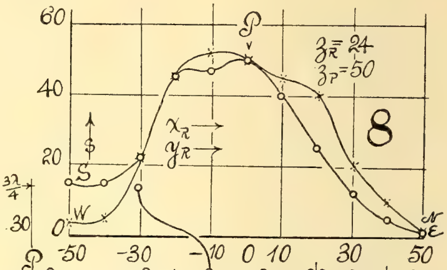
The resonator was finally transfered to a level, $\lambda/2 = 24$ cm. above the table and moved around the pipe axis, its mouth lying on successive circumferences of radii r, with the results given in figure 5. The maxima of all the curves are here curiously flattened; but there is still a gradual transfer of the chief crest from S to W, as r increases. Probably intermediate observations would have modified the curves. Since $\lambda/2$ determines a nodal plane normally below the pipe, the intensities are all relatively large, but they fall off as a whole so rapidly with r, that there is no intersection and all curves may be given in a single diagram.

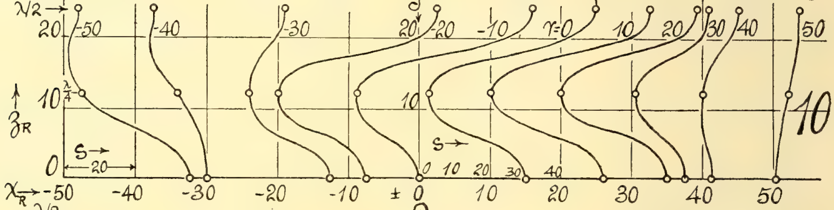
The graphs showing the intensity, s, along the S-N and W-E directions in figure 8, exhibit a total change of form. The two crests have seemingly all but coalesced near the pipe normal. Owing to an accident which somewhat dislocated the fringes, the indentation in the middle of the S-N curve and the bulge toward the right in the other may need modification but this is of little consequence here, since the plane at a level of $z = 24$; above the table contains a single marked crest, only.

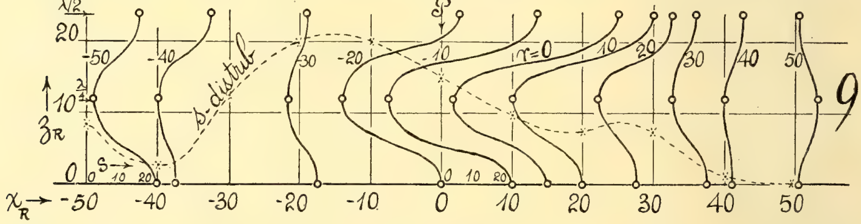
2. *Vertical Survey.*—It is now of interest to bring the values of intensity together when the resonator moves up from the table on the same vertical. As constructed in figure 9, with s laid off horizontally to the right, z_R vertically and with the origin of *each* tripoint s-curve at the x position of the vertical to which it belongs, these vertical distributions of nodal intensity strike the eye. The intensities s on the table for the same x positions are also sketched in. Under the pipe at $z_P = 50$ cm., the nodal surface at the table rapidly decreases in s-intensity to the antinode at $z_R = \lambda/4$; s then rapidly increases again to the more pronounced nodal surface at $z_R = \lambda/2$, nearer the pipe. Toward the south however the distinction between node and antinode is not regularly sustained: at $x = -20$ cm. the antinode has gained in strength and at $x = -30$ cm. the strength of nodes and antinodes does not differ much. Beyond this ($x = -40, -50$), however, nodes and antinodes are again sharply contrasted. Toward the north from the pipe normal ($x = 0$), these differences die out more gradually, until at $x = 50$ cm. the intensities are for some further reason, reversed.

What is very surprising, however, is the observation that throughout the whole of the extent of a linear meter ($x = \pm 50$) symmetrically to the pipe normal, the nodal and antinodal planes at $z_R = 0, \lambda/4, \lambda/2$, remain parallel to the table, so far as can be seen. The only distinction within this stretch is the distribution of nodal intensity as exhibited in the continuous curves sketched in, or separately in figures 6, 7 and 8. It would therefore here seem to be a mistake to associate wave-length with crests and troughs in these diagrams, however plausibly such inferences may obtrude. It is very much more probable that in the south some extraneous *s*-distribution has been superposed.









The West-East distributions (± *y*) though not so sharp in their features in figures 6, 7 and 8, show an equally pronounced character when treated in the same way relative to vertical variations, in figure 10.

Each of the triplet graphs is to be referred to an origin at the x_R marked on the curve and the *s*-intervals are 20 scale-parts, as heretofore. Since the edges of the table are at $y = \pm 60$ cm. (nearly), the nodal plane does not extend quite so far; but it does not follow that these edges have much influence on the results, since the same kind of graphs appear in figure 9, where the edges of the table are nearly 50 cm. further. The degradation of the nodal plane occurs first at $y = 50$ cm. in the east as it did in the north.

In the west it is still strong at $y = -40$, and now promises to run through another cycle of intensity as in the previous case. The antinode very nearly $s = 0$ throughout the east, begins to lose identity between $y = -20$ and -30, to be followed by oscillations. Finally, the first nodal plane above the table degenerates here more obviously than before, between $y = -30$ and -40.

The records in figures 9 and 10 contain an almost complete example of a free nodal region sustaining itself $z_R = 24$ cm. above the table, and figure 8 gives the acoustic pressures within it. These are largest near the pipe, though not quite symmetric and it is possible that the direct ray may here influence the resonator. In general, however, i.e., at greater distances, the direct ray is relatively ineffective and it is to the occurrence of nodal regions alone that the resonator then responds, even when the direct ray would seem to have the advantage of nearness. Thus in figure 10, the curve for $r = 0$, when tested at the second antinodal plane ($z_R = 36$ cm.), although the pipe is now but 12 cm. away, nevertheless retains the features of a pronounced minimum.

* Advance note from a Report of the Carnegie Inst., of Washington, D. C.

IONIZATION AND ABSORPTION EFFECTS IN THE ELECTRIC FURNACE

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In 1914, the publication by Fowler of the series formulae for enhanced lines showed that the Rydberg constant "N," occurring in the formulae for arc lines, is changed to 4N for enhanced lines. This established the view based on a very general difference of behavior in both laboratory and celestial sources, that the enhanced lines and the arc lines arise from differently constituted emitting centers. The condition pictured by Bohr's theory is that the enhanced lines are produced by atoms which have lost one electron, while the normal atom emits the arc lines.

Recent work by Saha applies the relations recognized in physical chemistry to the case when the normal atom is changed into an ionized atom with an electron set free. A definite equilibrium is considered as established, represented in the case of calcium by $Ca \rightleftarrows Ca^+ + e$.

It follows from this that if we add a large supply of electrons from some substance which becomes ionized more readily than calcium, fewer electrons from the calcium can exist in the free state, and the proportion of Ca^+ atoms becomes less, with a corresponding weakening of the enhanced lines relative to the lines of the normal atom.

In the electric furnace, the enhanced lines of some substances can be produced faintly, and the effect of a mixture with an easily ionized substance may be tested, under the same conditions of temperature and pressure for the mixture as for the pure substance. Experiments have been carried out in which both emission and absorption spectra were compared in this way.

The spectra of calcium and strontium showed in each case a decided weakening of the enhanced lines when a mixture of either element with potassium or caesium was examined, while the arc lines, presumably due to the unionized atoms, were unaffected. Barium, with a smaller difference between its ionizing potential and that of the added substance, showed a less pronounced effect. Sodium, whose ionization potential is nearly the same as that of barium, produced no weakening of the barium enhanced lines. The experiments thus support the view of an equilibrium between electrons and ionized atoms, which may be disturbed by an added supply of electrons.

Certain features of absorption spectra may be described briefly. By holding the furnace at a measured temperature and passing through the tube a beam of white light from a tungsten filament the absorption spectrum may be readily produced. Turning off the lamp enables the emission spectrum of the vapor at the same temperature to be photographed for comparison.

The iron spectrum under these conditons showed no decided difference in emission and absorption except as to the production of the ultra-violet spectrum. The low temperature stage (1600°–1700° C.) ended in emission at λ 3440, except that a few scattered lines as far as λ 3000 could be brought out faintly. The continuous spectrum when a plug is used in the tube at the same temperature has about the same limit, and it has been a general experience that the emission of vapors in the furnace extends no farther than the spectrum of a black body at the same temperature.

The possibility remained that metallic vapor too cool to emit in the far ultra-violet could still show absorption lines if a sufficiently hot source of white light in a chamber transparent to the ultra-violet were available. Lacking such a source, the light from exploding wires was tried, which Anderson has shown to give a continuous spectrum extending to short wave-lengths. Dr. Anderson kindly operated the explosion apparatus for this experiment, the best results being obtained with wires of lead. With the furnace temperature as low as 1600° C., absorption lines of iron could be identified as far as λ 2298, a limit set by the sensitiveness of the photograhic emulsion. These lines had previously been obtained as far as λ 2447 in the high temperature furnace when the continuous spectrum was supplied by a graphite plug in the tube. We thus have the condition that for the emission spectrum a definite limit is set by the black body radiation at that temperature, but that the vapor is able to absorb radiation of much shorter wave-length.

One feature of Saha's theory is that a vapor too cool to emit should be able to absorb the lines of the principal series. This was tried with the strong flame lines λλ 4227, 4607, and 5535 of calcium, strontium, and barium, respectively. In each case a temperature was found at which no emission could be detected, but when white light was sent through the furnace the lines appeared distinctly in absorption. For these favorable lines, the view of Saha is thus confirmed. It will be of interest to test other series lines in the same way.

In previous investigations on the absorption of vapors of the alkali metals, the lines of the principal series have appeared, but not those of the subordinate series. Since in emission spectra the latter require higher temperatures than the principal series lines, it seemed probable that a relatively high temperature was required to produce them in absorption. Experiment showed this to be the case. Sodium, potassium, caesium, and rubidium were tested by heating to a high temperature in a tube having a plug to give the continuous ground. The subordinate series lines appeared in absorption for each element.

The general character of the absorption phenomena in the furnace may be described as in general a reversal, when white light is passed through the tube, of the emission spectrum at the same temperature. Exceptions are found in the easily excited principal series lines and perhaps others, which the vapor can absorb before it emits, and in the ultra-violet region beyond the limit, at the given temperature, of the black-body radiation. In the latter case an absorption spectrum appears beyond the emission limit when a sufficiently hot source of white light is employed.

PRELIMINARY NOTE ON A BIOMETRICAL STUDY OF THE RELATIONS OF CERTAIN VISCERA IN TUBERCULOSIS[1]

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One of the most fundamental problems of biology is that of adaptive regulation to the end of maintaining the life of the individual. The ability to readjust the functional relations of the parts of the organism, after they have been in any way disturbed is widespread among living things generally, including man. A well-known example is seen in the functional hypertrophy of the kidney. If one kidney is removed the remaining one promptly enlarges and carries on the work formerly divided between the two. Many similar examples might be cited.

We have been engaged for the past year in a study of this phenomenon of adaptive functional regulation from a somewhat new point of view. Starting from the justifiable assumption that the records of pathological anatomy, as set down in autopsy protocols, should if adequately analyzed give a picture of regulatory processes perhaps more detailed and definite than is likely to be found elsewhere, it seemed probable that quantitative study of the ratios existing between the weights of the more important viscera would indicate with great exactness any disturbances in the normal balance of the parts of the organism. The results, we believe, fully bear out this assumption. This first investigation, which has just been completed, was restricted entirely to tuberculosis as the pathological condition involved. This choice was deliberate and based upon the known profound constitutional effects of the disease. The data studied are six organ-weight indices or ratios, defined as follows:

$$\text{Index A} = \frac{\text{Liver weight}}{\text{Heart weight}} = L/H$$

$$\text{Index B} = \frac{\text{Liver weight}}{\text{Spleen weight}} = L/S$$

$$\text{Index C} = \frac{\text{Liver weight}}{\text{Right kidney weight} + \text{Left kidney weight}} = L/K$$

$$\text{Index D} = \frac{\text{Heart weight}}{\text{Spleen weight}} = H/S$$

$$\text{Index E} = \frac{\text{Right kidney weight} + \text{Left kidney weight}}{\text{Heart weight}} = K/H$$

$$\text{Index F} = \frac{\text{Right kidney weight} + \text{Left kidney weight}}{\text{Spleen weight}} = K/S$$

These indices were calculated from the data given on the protocols of such of the first 5000 autopsies (1341 in all) of the Johns Hopkins Hospital as revealed tuberculous lesions, and at the same time conformed to certain other requirements. The material was divided for purposes of study by race, sex, activity or quiescence of lesion at time of death, age, and character of lesion.

The age characteristics of this autopsied, tuberculous, hospital population were compared in detail with those of the general population, and with that part of it dying of tuberculosis. In general there is a close agreement in mean age at death between the autopsied hospital population showing active tuberculous lesions, and the portion of the general population dying of tuberculosis.

One of the most interesting and novel results of the study has been to show that certain of these organ-weight indices change with the age of the individual. These changes may be looked upon as phenomena of senescence beginning very early in life, indeed practically at birth. The detailed results as to correlation with age may be briefly summarized as follows. There is a significant *negative* correlation in all groups between the value of Index A and age. This means that as age advances the liver/heart ratio tends to take on lower and lower values. Index B is, on the whole not significantly correlated with age. The same is also true of Index C. With the exception of the white groups with active tuberculous lesions, which show no sensible correlations of this index with age, there is generally a significant *positive* correlation of Index D with age. Index E exhibits a relatively high, and in general certainly significant *negative* correlation with age. Index F is not correlated with age. Inasmuch as the material included all ages from infancy to extreme old age, these correlational results with age have a general biological significance. They indicate that, so far as the present material may be trusted to portray general relationships, the functional balance of certain organs, notably the heart and spleen, changes throughout life in an orderly manner, capable of expression by mathematical equations. Such regression equations, which are throughout linear in character, were computed for all cases where the correlation coefficients were significant, and made the basis of later corrections of the biometric constants of the indices, to allow for the influence of age.

A detailed study of the relation of race (white or colored) to the values of the several organ weight indices, after proper corrections had been made for the influence of age, leads to the general result that where there are no lesions in either of the organs involved in an index and where the number of cases is large enough to give reliable results, there is, broadly speaking, no difference between white and colored racial groups in either mean or variability, except in the case of Indices B and D. In these cases it is only the means and not the relative variabilities that differ. In both cases it is the colored group that has the higher value for the indices.

In general there are no significant differences between the sexes in either mean values or variability of the indices, after proper corrections have been made for the influence of age. In these organ-weight indices we are evidently dealing with fundamental functional characteristics of the organism, which express in a hitherto unnoted way the extraordinary regulatory powers which are in a profound manner associated with the maintenance of life. The absolute weights of the several viscera may differ widely in the different races and sexes, but the ratios of these same viscera, in respect of weight, appear from the present experience to have a constancy and stability biometrically, which, broadly speaking, entirely transcends the influence of race and sex.

After making proper corrections for differences in the age distributions of the groups involved, the question of the influence of tuberculosis *per se* upon the organ-weight indices was attacked. This problem was approached in several ways of which the most significant was to compare two groups of which one contained only cases wherein the *sole* significant lesions at death were those of tuberculosis, while the other contained cases where in addition to the tuberculous lesions present (active or inactive) were *other* lesions in themselves sufficiently grave to have caused death, had there been no associated tuberculosis. On the basis of this grouping it is found that five of the six indices (namely all those involving heart or spleen or both) show large and significant differences as between the two groups. These differences everywhere are of the sort which would arise if the effect of fatal tuberculosis was to lower the absolute weight of the heart and increase that of the spleen. The significance of these results, as well as that of the differences in mean age at death in the two groups will be discussed in the detailed paper, which will appear in the Reports of the Johns Hopkins Hospital.

The work is being continued with other pathological groups, and it is believed will become more interesting and significant as we become able to compare the results from different types of lesions.

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[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 58.

THE EFFECT OF ETHER UPON THE MIGRATION OF THE SCALE PIGMENT AND THE RETINAL PIGMENT IN THE FISH, FUNDULUS HETEROCLITUS

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When an adult Fundulus is etherized the pigment in the scale melanophores shows a complete distal migration and the fish becomes dark in color. In this respect ether produces an effect the reverse of adrenalin. But adrenalin not only causes a proximal migration of the scale pigment; it induces a distal migration of the retinal pigment (Gilson, '22). Does ether also influence the retinal pigment and, if so, is its effect on that pigment the reverse of that of adrenalin? To test this question Fundulus was etherized either in the dark or in the light and its eyes prepared for study. Fish that had been some time in the light were etherized by allowing a stream of 5% ether to drip upon the gills from ten to fifteen minutes. Before etherization the scale melanophores showed a complete proximal migration of the pigment granules. Within one to four minutes after the application of ether had begun the melanophores were in a state of complete distal migration. At the end of ten or fifteen minutes the animals were killed and the eyes were prepared. The same procedure was carried out upon fish that had been kept in the dark. 5% ether was also applied to both light and dark fish by immersing the body as far as the gills in the ether solution for twenty or thirty minutes after which their eyes were prepared. The scale melanophores of light fish which were immersed in this way showed a distal migration in from five to fifteen minutes wherever the ether solution touched the skin. As controls in each experiment eyes of unetherized fish that had been kept either in the light or in the dark were prepared.

The condition of the retinal melanophores of fish which had been etherized, either by applying the solution to the gills or to the body, differed in no way from those of normal fish kept under the same conditions of illumination. Those of light fish showed a distal and those of dark fish a proximal migration of the pigment granules. Ether was applied directly to the eyes by immersing excised eyes in a 5% solution for thirty minutes followed by immediate fixation. Very commonly the pigment in eyes which were treated in this way was extremely contracted and broken up into rather large round masses separated from each other by spaces much larger than any seen in normal eyes. This was probably a post mortem change due to the disintegration of the pigment cells. In eyes which appeared normal, however, the condition of the pigment was not different from that of retinas which had not been treated with ether. The post mortem change described above was often seen in eyes whose optic nerves had been cut before the application of ether. Here too the pigment cells did not seem to be affected at all by the ether.

Arey ('16) found that carbon dioxide and ether, both in darkness and in light, and in excised as well as in undisturbed eyes, completely check the movement of all the retinal elements in fishes. The retinal pigment of fishes anesthetized under one condition of illumination and then removed to another remained in the condition characteristic of the first state of light or darkness. Hence I conclude that ether does not have an action on the pigment cells of the retina in fishes the reverse of that of adrenalin. It merely arrests the retinal pigment in whatever condition it happens to be when the drug is administered.

An explanation for the differing effects of adrenalin and ether upon the scale and retinal melanophores in fishes may probably be found in the widely different methods of control of the two types of melanophores. The scale melanophores are controlled by the sympathetic nervous system and in the light they are held in the contracted condition by means of a tonus established by impulses in the central nervous system set up by light stimuli received through the eyes. The contracted state, then represents a state of stimulation. Spaeth ('16) thinks that melanophores are a physiologically modified type of smooth muscle tissue. Adrenalin, the effect of which upon smooth muscle tissue is comparable to sympathetic stimuli, causes a contraction of scale melanophores. Ether anesthetization removes the nervous tonus and allows the melanophores to expand. The melanophores of the retina, however, are not controlled by means of 'retino-motor' nerve fibres (Arey, '16, p. 180), but their activities are simply "expressions of protoplasmic responses to definite stimulating agents." The effect of adrenalin is the expression of the action of such an agent from the blood. Ether inhibits the activities of the melanophores arresting them in whatever condition they may happen to be.

Arey, L. B., 1916. The movements in the visual cells and retinal pigment of the lower vertebrates. *J. Comp. Neurol.*, **26** (121–190).

Gilson, A. S., 1922. The diverse effects of adrenalin upon the migration of the scale pigment and the retinal pigment in the fish, Fundulus heteroclitus, Linn. *Proc. Nat. Acad. Sci.*, **8** (130–133).

Spaeth, R. A., 1916. Evidence proving the melanophore to be a disguised type of smooth muscle cell. *J. Exp. Zoöl.*, **20** (193–215).

THE DIVERSE EFFECTS OF ADRENALIN UPON THE MIGRATION OF THE SCALE PIGMENT AND THE RETINAL PIGMENT IN THE FISH, FUNDULUS HETEROCLITUS, LINN

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Bigney ('19), working with the frog, found that the injection into the blood stream of this animal of small quantities of adrenalin caused a proximal migration (1) of the granules in the dermal melanophores of the skin and a distal migration of the granules in the melanophores of the retina. The present investigation was undertaken to determine if similar effects were to be found in fishes.

For these experiments, the animal used was the common kilifish, Fundulus heteroclitus Linn. This fish shows a marked response, both in the scale (dermal) and the retinal melanophores to light and to darkness. When it is placed in the light over a white background, it shows a proximal migration of the scale melanophore granules and a distal migration of the retinal melanophore granules. In the dark, these conditions are reversed, the scale melanophores showing a distal and the retinal melanophores a proximal migration.

The method adopted for stating the effects of the adrenalin upon the retinal pigment was that of expressing the width of the retinal pigment extension, as measured along processes showing maximal extension in the region measured, as a fraction of the total distance from the outer boundary of the pigmented epithelium to the external limiting membrane, this distance being taken as 100. For purposes of brevity in this paper, the terms *light fish* and *dark fish* will be used to indicate fish which have been kept in the light and fish which have been kept in the dark, respectively. In the typical light fish, the pigment shows an average extension of 83 units on this scale. Such light retinas show, in almost all cases, a secondary concentration of pigment between 70 and 80. In dark animals, the average extension of the retinal pigment is 59, the pigment being massed at the outer region of the retina.

Retinas were examined of normal light fish, of normal dark fish, and of light and dark fish some of which had received intramuscular, some intra-abdominal injections of physiological salt solution, of adrenalin 1:1,000, or of adrenalin 1:10,000. The adrenalin used was that prepared by Parke Davis and Company and sold as "Adrenalin 1:1,000." The adrenalin was allowed to act for varying times before the retinas were fixed, with the results that the greatest effect was found to occur in those dark animals which had been injected with 0.2 cc. of adrenalin 1:1,000, and between thirty minutes and one hour after injection.

Of the animals to be experimented upon, those to be used as dark animals were kept in a photographic darkroom for at least sixteen hours before being treated; those to be used as light animals were placed in a white dish which was kept for at least a day by a north or west window and the fish were treated and killed between one and three o'clock in the afternoon. The animals were killed by quick decapitation with a pair of scissors, the head was opened longitudinally, and immersed in Bouin's picro-formol mixture. Fixation was allowed to proceed for about fifteen hours. The eyes were then dissected free from the head, washed thoroughly in 70% alcohol and dehydrated in 90% and absolute alcohol. While in absolute alcohol, the cornea of each eye was sliced off with a razor and the lens removed with a pair of fine forceps. The eyes were then cleared in xylol, infiltrated and imbedded in paraffin and serial sections were cut, in most cases 7 micra in thickness, and in a plane perpendicular to the face of the eye and through the optic nerve. Measurements could thus be made on radial sections at a point about half way from the entrance of the nerve to the outer rim of the retina. For measuring, a high dry objective and camera lucida were used, projection being made upon a line ruled along a sheet of paper and upon which the actual measurement was made. All experiments were performed at room temperature.

The results obtained are summarized in *Table A*. The values given in the table are the averages of all experiments for each class, plus or minus the probable error (P. E.) as obtained by the use of Bessel's formula. This probable error indicates the relative value of the results by taking into account not only the standard variation of each series but also the number of experiments performed.

It will be seen that when light animals are injected with 0.2 cc. of adrenalin 1 : 1,000, there is no *significant* change in the width of the pigmented region of the retina. In dark animals which had been injected into the body cavity with 0.2 cc. of adrenalin at the strength of 1:1,000, there was shown a marked change of the pigment position, the pigment

TABLE A.

Distance from outer boundary of pigmented epithelium to external limiting membrane	100
Extent of retinal pigment processes in:	
Normal Light Fish	82 ± 2
Normal Dark Fish	59 ± 3
Light Fish, 0.2 cc. Adrenalin 1 : 1,000 intra-abdominally	80 ± 1
Dark Fish, 0.2 cc. Physiological salt solution, intra-abdominally	64 ± 1
Dark Fish, 0.2 cc. Adrenalin 1 : 1,000 intra-abdominally	77 ± 2
Dark Fish, 0.2 cc. Adrenalin 1 : 10,000 intra-abdominally	66 ± 3
Dark Fish, 0.2 cc. Adrenalin 1 : 1,000 intra-muscularly	74 ± 11
Dark Fish, 0.2 cc. Adrenalin 1 : 10,000 intra-muscularly	68 ± 3

Average weight of fish = 4.3 grams

$$\text{P. E.} = 0.6745 \frac{\Sigma\, d^2}{n(n-1)}$$

where d is the deviation of any observation from the mean and n is the number of experiments in the series.

processes having an average extension of 77. Qualitative examination of these retinas showed a marked tendency toward the light condition. In these animals, however, the secondary, distal concentration of pigment was usually lacking. Injection into the abdominal cavity of weaker adrenalin solution, and injection into the dorsal musculature gave generally unsatisfactory results. Dark animals into whose body cavity physiological salt solution had been injected showed a barely significant change in the condition of the retinal pigment. Whether this very slight change is due to the effect of the physiological salt solution itself, or to the secretion of adrenalin by the animal remains to be determined. In all cases, the scale melanophores of animals injected with adrenalin showed the extreme proximal migration of the melanophore granules which has been described by previous workers.

Klett ('08), working with the frog, and using only light animals, was able to discover no effect upon the retinal pigment when a solution of adrenalin was injected into the blood stream. Injection of a very strong solution of adrenalin into the anterior chamber of the eye, however, caused a contraction of these melanophores. Bigney believes, and probably rightly, that this effect was due to the toxic nature of the solution used. Fujita ('11) used dark frogs and found that the injection of adrenalin into the bloodstream caused a distal migration of the retinal pigment. This finding was confirmed by Bigney.

Experiments upon Fundulus, here reported, demonstrate that the injection of adrenalin will cause a distal migration of the pigment processes in the retinas of dark animals, but produces no significant change in the already extended processes of light animals. The changes are of a reverse nature in the scale melanophores, the injection of adrenalin into the animal causing a marked proximal migration of the granules. These findings are thus in accord with those of Fujita and Bigney in their work upon the frog, and opposed to those of Klett, also upon the frog.

Arey, L. B., 1916. "The Movements in the Visual Cells and Retinal Pigment of the Lower Vertebrates." *J. Comp. Neur.*, **26** (121–190).

Bigney, A. J., 1919. "The Effect of Adrenin on the Pigment Migration in the Melanophores of the Skin and in the Pigment Cells of the Retina of the Frog." *J. Exp. Zoöl.*, **27** (391–396).

Fujita, H., 1911. "Pigmentbewegung und Zapfenkontraktion im Dunkelauge des Frosches bei Einwirkung verschiedener Reize." *Arch. vergl. Ophthalm.*, Jahrg. **2** (164–179).

Klett, 1908. "Zur Beeinflussung der phototropen Epithelreaktion in der Froschretina durch Adrenalin." *Arch. Anat. Physiol.*, Jahrg., 1908, *Physiol. Abt., Suppl. Bd.* (213–218).

[1] Arey ('16) used the terms *distal* and *proximal* to designate the migration of the pigment granules away from the center of the melanophore cell and toward that center respectively. Hence a proximal migration would bring about what is ordinarily called the contracted condition of the melanophore and a distal migration the expanded condition.

A SIDE LINE IN THE IMPORTATION OF INSECT PARASITES OF INJURIOUS INSECTS FROM ONE COUNTRY TO ANOTHER

By L. O. Howard

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Since the extraordinary initial success of the Department of Agriculture in importing the Australian ladybird beetle (*Novius cardinalis*) into California to destroy the fluted scale in the eighties, a great deal of successful work of the same sort has been done in different parts of the world. Most of these attempts have been made in a haphazard sort of way and a number of them have succeeded to a notable degree in spite of the rather unscientific way in which the importations were made and the lack of competent scientific study before the importations were attempted. The one thing which seems to have been rather carefully guarded from the start is the possibility of importing secondary parasites which might destroy the desirable assisted immigrants. Until recently there have seemed to be no time and no especial necessity for a careful biological study of the imported parasites before the importation has been attempted. Now in many cases such preliminary studies seem to be very necessary and they are being made.

One point which has undoubtedly resulted in the non-establishment of imported species has been the fact that in their native homes they had alternate hosts, feeding during one generation upon one species and during another generation upon another. Frequently the alternate host or a vicarious species has not existed in the country into which the parasites have been introduced.

On the other hand, such secondary and supplementary hosts have in a number of cases existed there, and as a result the establishment of the imported species was not only rendered more certain but the species has acted in a most beneficial way by destroying other pests, some of them in fact being native to the country into which the imported parasite was brought.

It is this side line in such importations upon which I wish to dwell for a moment in this paper.

Parasite introduction has never been attempted in any part of the world on so large a scale as it has been in this country since 1905 in the effort to secure the European and Japanese parasites and natural enemies of the gipsy moth and the European enemies of the brown-tail moth. In this work, although more than thirty species have been imported, not more than seven or eight have become established. Lack of secondary hosts may have been the cause of the failure of many of them, or there may have been other causes. An attempt is being made at the present time to determine these causes, and for the first time since the world war experts from the Bureau of Entomology are now in Europe and Japan studying the native parasites of the gipsy moth and endeavoring to send over new supplies of those species which previously failed of establishment and at the same time to secure additional species which we had not found before the war.

But the value of the comparatively few species that have actually become a part of the insect fauna of the United States as the result of the earlier introductions has been very considerably enhanced from the fact that some of them have taken readily to other imported or native caterpillars which destroy American trees and crops. The following table indicates the present situation with regard to four of these species.

HYMENOPTEROUS PARASITES

HOST FOR WHICH IMPORTED	NAME OF PARASITE	AMERICAN SPECIES ATTACKED	COMMENT ON ABUNDANCE
Euproctis chrysorrhoea L.	*Apanteles lacteicolor* Vier.	*Datana ministra* Drury	Rare
	Apanteles lacteicolor Vier.	*Hyphantria textor* Harris	Rare
	Apanteles lacteicolor Vier.	*Apatela hasta* Guenée	Rare
	Apanteles lacteicolor Vier.	*Schizura unicornis* S. & A.	Only in the laboratory, but here readily attacked
Euproctis chrysorrhoea L.	*Meteorus versicolor* Wesm.	*Hemerocampa leucostigma* S. & A.	Rare
	Meteorus versicolor Wesm.	*Hyphantria textor* Harris	Rare
Porthetria dispar L.	*Apanteles melanoscelus* Ratz.	*Hemerocampa leucostigma* S. & A.	Apparently abundant

DIPTEROUS PARASITES

HOST FOR WHICH IMPORTED	NAME OF PARASITE	AMERICAN SPECIES ATTACKED	COMMENT ON ABUNDANCE
Porthetria dispar L., and *Euproctis chrysorrhoea* L.	**Compsilura concinnata* Meig.	*Stilpnotia salicis* L.	Common and important
	**Compsilura concinnata* Meig.	*Basilarchia archippus* Cramer	Common and important
	**Compsilura concinnata* Meig.	*Euvanessa antiopa* L.	Common and important
	**Compsilura concinnata* Meig.	*Vanessa atalanta* L.	Common and important
	**Compsilura concinnata* Meig.	*Aglais milberti* God.	Common and important
	**Compsilura concinnata* Meig.	*Diacrisia virginica* Fab.	Common and important
	**Compsilura concinnata* Meig.	*Datana ministra* Drury	Common and important
	**Compsilura concinnata* Meig.	*Euchaetias egle* Drury	Common and important
	**Compsilura concinnata* Meig.	*Estigmene acraea* Drury	Common and important
	**Compsilura concinnata* Meig.	*Hemerocampaleucostigma* S. & A.	Common and important
	**Compsilura concinnata* Meig.	*Cirphis unipuncta* Haw.	Common and important
	**Compsilura concinnata* Meig.	*Papilio polyxenes* L.	Common and important
	**Compsilura concinnata* Meig.	*Papilio turnus* L.	Common and important
	**Compsilura concinnata* Meig.	*Anosia plexippus* L.	Common and important
	**Compsilura concinnata* Meig.	*Apatela hasta* Guenée	Common and important
Porthetria dispar L., and *Euproctis chrysorrhoea* L.	**Compsilura concinnata* Meig.	*Apatela brumosa* Guenée	Common and important
	**Compsilura concinnata* Meig.	*Schizura concinna* S. & A.	Common and important
	**Compsilura concinnata* Meig.	*Polygonia interrogationis* Fab.	Common and important
	**Compsilura concinnata* Meig.	*Papilio troilus* L.	Common and important
	**Compsilura concinnata* Meig.	*Alypia octomaculata* Fab.	Common and important
	**Compsilura concinnata* Meig.	*Epargyreus tityrus* Fab.	Common and important
	**Compsilura concinnata* Meig.	*Mamestra adjuncta* Boisd.	Apparently important

*Besides the above listed hosts we have reared *Compsilura* from about 60 other species of native larvae. We also have records of several more species reared by others.

Dipterous Parasites (Cont'd)

HOST FOR WHICH IMPORTED	NAME OF PARASITE	AMERICAN SPECIES ATTACKED	COMMENT ON ABUNDANCE
	**Compsilura concinnata* Meig.	*Notolophus antiqua* L.	Apparently important
		Croesus latitarsus Nort.	Apparently important
	**Compsilura concinnata* Meig.	*Phlegethontius quinquemaculata* Haw.	Apparently important
		Pontia rapae L.	Apparently important
	**Compsilura concinnata* Meig.	*Malacosoma americana* Fab.	Rare
	**Compsilura concinnata* Meig.	*Malacosoma disstria* Hübn.	Rare
	**Compsilura concinnata* Meig.	*Hyphantria textor* Harris	Rare
	**Compsilura concinnata* Meig.	*Mamestra picta* Grote	Rare
	**Compsilura concinnata* Meig.	*Halisidota caryae* Harris	Rare
	**Compsilura concinnata* Meig.	*Halisidota tessellaris* S. & A.	Rare
Porthetria dispar L., and *Euproctis chrysorrhoea* L.	**Compsilura concinnata* Meig.	*Melalopha inclusa* Hübn.	Rare

Predators

HOST FOR WHICH IMPORTED	NAME OF PARASITE	AMERICAN SPECIES ATTACKED	COMMENT ON ABUNDANCE
Porthetria dispar L., and *Euproctis chrysorrhoea* L.	*Calosoma sycophanta* L.	*Porthetria dispar* L.	Abundant in sections of New England where oak trees abound that are heavily infested with this host. *Important enemy*
	Calosoma sycophanta L.	*Euproctis chrysorrhoea* L.	Common
	Calosoma sycophanta L.	*Stilpnotia salicis* L.	Apparently important
	Calosoma sycophanta L.	*Heterocampa guttivitta* Walk.	Rare
	Calosoma sycophanta L.	*Hemerocampa leucostigma* S. & A.	Rare
	Calosoma sycophanta L.	*Ennomos subsignarius* Hübn.	Common
	Calosoma sycophanta L.	Noctuid sp. (Tree climbing larvae.)	Common

PREDATORS (Cont'd)

HOST FOR WHICH IMPORTED	NAME OF PARASITE	AMERICAN SPECIES ATTACKED	COMMENT ON ABUNDANCE
	Calosoma sycophanta L.	*Malacosoma americana* Fab.	Rare
	Calosoma sycophanta L.	*Malacosoma disstria* Hübn.	Common
	Calosoma sycophanta L.	*Geometrid* sp.	Common

This predator does not refuse or hesitate to attack any species of Lepidopterous larvae ordinarily met on tree growth during the summer months at the time of its abundance; hence there are many other species that fall prey to it, the names of which are not yet recorded. Several species of Lepidopterous larvae have been offered as food to the beetles in jars and cages which were readily consumed by them but are not recorded above.

PREDATORS (Cont'd)

HOST FOR WHICH IMPORTED	NAME OF PARASITE	AMERICAN SPECIES ATTACKED	COMMENT ON ABUNDANCE
Porthetria dispar L., and *Euproctis chrysorrhoea* L.	*Carabus auratus* L.	Gray field slug (*Agriolimax agrestis* L.)	Apparently important
	Carabus auratus L.	Noctuid sp. (Ground cutworms)	Apparently important

This Carabid undoubtedly devours other species of soft-bodied insects ordinarily found in or near the ground.

Of these, it will be seen that the Tachinid fly *Compsilura concinnata* has readily accommodated itself to a number of native injurious species and that it is an introduction which is of great importance entirely aside from its efficiency as a parasite of the brown-tail moth.

It seems reasonably certain also that the little Braconid *Apanteles lacteicolor* is a species which will prove to have a very broad value. In addition to the records already given, I am informed by Mr. R. T. Webber of the Gipsy Moth Laboratory that the Tachinid mentioned is also probably a parasite on the larva of *Cimbex americana* (a sawfly) and on two other, unidentified sawflies; while Mr. C. W. Johnson, Curator of the Boston Society of Natural History, has noted an apparent parasitism of this species on the pine weevil (*Pissodes strobi*).

A most useful feature in the biology of the Tachinid fly is that with the larger caterpillar hosts the third stage of growth is the one in which they are nearly always attacked, which allows their Microgaster parasites to develop and emerge without having to struggle against a simultaneous parasitism by the Tachinid.

Moreover, this important imported Tachinid has not only spread all over the whole gipsy moth territory in the United States but is found in some cases twenty miles beyond the line of gipsy moth spread.

I confess that I anticipate great benefits in this same broad way from *Apanteles lacteicolor* since it apparently attacks not only hairy caterpillars but naked caterpillars.

Of late the Bureau of Entomology has been introducing from south Europe the parasites of the imported European corn borer. Although it is too early to state positively that any of these imported parasites have been established, an important parasite known as *Habrobracon brevicornis*, found last summer in a region of south France near Hyères, has been brought over in very considerable numbers and has been reared in the laboratory with the utmost ease and some thousands have been liberated. It manages to lay its eggs in the larva of the borer while in the cornstalk, and, as it happens that we have a native cornstalk borer which is especially prevalent in the southeastern United States, we have experimented in the laboratory at Arlington, Massachusetts, with the latter species, and find that the European parasite takes to it readily and should become, if we can establish it, an important parasite of the southern corn pest. This work was done on the initiative of and under the direction of Mr. W. R. Walton.

Further than this, there exists a Lepidopterous borer in the sugar cane in Louisiana which is closely allied to the corn stalk-borer of the southern States and which will with little question be parasitized by this European parasite when they are once brought together. Inasmuch as the Habrobracon normally inhabits the Mediterranean littoral, it is altogether likely that it will accommodate itself to the climate of the Gulf of Mexico littoral more readily than it will the harsher climate of New England and New York. The experiment has actually been begun and it is found that the European parasite breeds freely in confinement at New Orleans on the sugar-cane borer. It is planned to liberate large numbers the present season on an isolated infested plantation in Louisana. We have strong hopes of its successful establishment.

The evolution of parasitism with insects is an interesting study. As in so many other directions in biological studies, we can trace the course of probable evolution by different stages which exist at the present time. The very catholic methods of the Tachinid fly listed above, which lays its eggs not only on all sorts of caterpillars, whether hairy or naked or large or small, but will even oviposit on the caterpillar-like sawfly larvae, and even upon a weevil grub in one instance (not, however, as yet fully verified), indicate rather surely that the parasitic habit has more recently become adopted with the Tachinids than with very many of the parasitic Hymenoptera.

But with the Hymenoptera we have all stages in the evolution, from general parasitism to a highly specific parasitism, in the latter cases the parasite having become so perfectly modified to fit itself to the one species of host that it must mean an advanced stage of evolution.

In considering the importation of parasite species we must aim to have both kinds if they exist—the specifically constituted parasites and the general parasites. From the ranks of the latter we will have great help in the destruction of large numbers of the particular insect against which it was imported, but it will have the additional advantage of accommodating itself to other injurious species, and hence on the whole will probably be a more desirable addition to the American parasite fauna.

Although not parasites, mention should be made of two species of ground-beetles brought over from Europe in the course of the gipsy moth work. *Calosoma sycophanta* has been found to attack a number of injurious native caterpillars; and *Carabus auratus*, while probably primarily a feeder upon slugs, as it is in its native home, also feeds upon cutworms and probably other soft-bodied insects ordinarily found in or near the ground.

ON THE LOCATION OF THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL

By J. L. Walsh

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Communicated April 11, 1922

This note considers some geometric aspects of the problem of the approximate location of the roots of the derivative $f'(z)$ of a polynomial $f(z)$ when the roots of $f(z)$ are known. If $f(z)$ has m_1 roots in a circle C_1, m_2 roots in a circle C_2, and no other roots, the roots of $f'(z)$ are known to lie in C_1, C_2, and a certain third circle C which is readily determined.[1] Moreover, if $f(z)$ has m_1 roots in C_1, m_2 roots in C_2, m_3 roots in C_3, and no other roots, and if the circles C_1, C_2, C_3, are equal and have collinear centers, the roots of $f'(z)$ are known to lie in those circles and in two circles C', C'' equal to the original circles and whose centers are collinear with their centers.[2] In each of these cases, the actual *locus* of the roots of $f'(z)$ consists of the circles stated, when the given circles are the loci of those roots of $f(z)$ (supposed to vary independently) which they contain.

The second of these results suggests the problem of the determination of the locus of the roots of $f'(z)$ when the locus of the roots of $f(z)$ consists of three circles C_1, C_2, C_3 which are not supposed equal and with collinear centers.[3] The solution of that problem is indicated in the present note; the answer is stated in the

THEOREM. *Let circles C_1, C_2, C_3 be the respective loci of m_1, m_2, m_3 roots of a polynomial $f(z)$. Then the locus of the roots of $f'(z)$ consists of C_1, C_2,*

C_3 (except that C_i is to be omitted if $m_i = 1$) and of a region R bounded by the bicircular quartic

$$
\begin{aligned}
& m_1^2[(x-b_1)^2 + (y-b_2)^2-r_2^2]\,[(x-c_1)^2 + (y-c_2)^2-r_3^2] \\
& +m_2^2[(x-a_1)^2 + (y-a_2)^2-r_1^2]\,[(x-c_1)^2 + (y-c_2)^2-r_3^2] \quad \times \\
& +m_3^2[(x-a_1)^2 + (y-a_2)^2-r_1^2]\,[(x-b_1)^2 + (y-b_2)^2-r_2^2] \\
(1)\quad & +2m_2m_3[(x-a_1)^2 + (y-a_2)^2-r_1^2]\,[(x-b_1)\,(x-c_1) + (y-b_2(\,(y-c_2)-r_2r_3] \\
& +2m_1m_3[(x-b_1)^2 + (y-b_2)^2-r_2^2]\,[(x-a_1)\,(x-c_1) + (y-a_2)\,(y-c_2)-r_1r_3] \\
& 2m_1m_2[(x-c_1)^2 + (y-c_2)^2-r_3^2]\,[(x-a_1)(x-b_1) + (y-a_2)\,(y-b_2)-r_1r_2] = 0;
\end{aligned}
$$

here the circles C_1, C_2, C_3 have the respective centers and radii: $(a_1, a_2), r^1$; $(b_1, b_2), r_2$; $(c_1, c_2), r_3$. If the curve (1) consists of two nested ovals, R is the region bounded by the outer oval; if (1) does not consist of two nested ovals, R is the interior of the curve.

If no two of the regions C_1, C_2, C_3, R have any point in common, they contain respectively $m_1 - 1$, $m_2 - 1$, $m_3 - 1$, 2 roots of $f'(z)$. If R is in two parts, which have no point in common with each other or any of the regions C_i which is a part of the locus of the roots of the derivative, each part of R contains one root of $f'(z)$.

This theorem replaces the problem of the exact location of the roots of $f'(z)$, which depends on the solution of an equation of degree $m_1 + m_2 + m_3 - 1$, by the problem of their approximation which is solved by plotting a quartic curve, which depends on the solution of equations of degree 4. In particular cases—such as that mentioned in the first paragraph of this note—the quartic (1) degenerates, in which case the new problem is still simpler, for it depends merely on the solution of quadratics.

The theorem can be proved by a straightforward method which we now indicate. The locus of the roots of $f'(z)$ is readily found when C_1 is not a null-circle but C_2 and C_3 are both null-circles; every point of the boundary is a point of the curve (1) where $r_2 = r_3 = 0$. The locus of the roots of $f'(z)$ can then be found when neither C_1 nor C_2 is a null-circle by determining the envelope of the former boundary; every point of the new boundary is a point of (1) where $r_3 = 0$. Finally the locus is found in the general case by determining the envelope of this last determined boundary; every point of the boundary is a point of the curve (1).

A particularly interesting case of the theorem occurs if $m_1 = m_2 = m_3$, $r_1 = r_2 = r_3$, and the centers of C_1, C_2, C_3 are the vertices of an equilateral triangle. The locus of the roots of $f'(z)$ is (in addition to the regions C_i) the interior and boundary of a circle whose center is the center of this triangle and whose radius is $(r_1^2 + r_1h)^{1/2}$, h being the distance from the center of the triangle to the centers of the circles C_i.

The geometry of this entire situation deserves further inquiry, with especial reference to the cases where (1) degenerates. Our theorem generalizes presumably to the case in which any number of circles C_i form the loci of the roots of $f(z)$, and also where we consider not merely the first but the k^{th} derivative of $f(z)$. The writer hopes to continue the study of these problems.

[1] See paper by the writer, *Trans. Amer. Math. Soc.*, **22,** 1921 (101–116).

[2] See a forthcoming paper by the writer in the *Trans. Amer. Math. Soc.;* a similar result holds for any number of equal circles C_i which have collinear centers.

[3] This problem is set forth explicitly in the paper referred to in the preceding footnote.

ON THE DISTRIBUTION OF THE VELOCITIES OF STARS OF LATE TYPES OF SPECTRUM

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Read before the Academy April 25, 1922

The recent determination of spectroscopic paralaxes or stars of the later types of spectrum made by Adams and his collaborators[1] at the Mt. Wilson Observatory has made it possible to compute the absolute motions in space of about 1300 stars of spectral types A_7 to M for which the proper motions and radial velocities are known. The spectroscopic parallaxes for which the errors are proportional to the parallaxes themselves are in general better adapted for this purpose than the trigonometric parallaxes since the accuracy of the smaller parallaxes is considerably greater.

The three velocity-components of these stars have been computed with reference to the galactic system of coördinates, the pole of galaxy being assumed to have the position A = 190.6°; D = + 27.2°. This system of coördinates is very convenient in the study of the systematic motions of the stars since these motions are mainly in the galactic plane.

Solar Motion.—The solar motion can be obtained directly by forming the algebraic mean of the velocity-components which are measured with reference to the sun. By taking the opposite vector we find the sun's velocity relative to the group of stars employed. The following values for the sun's velocity have been derived in this way.

SPECIAL TYPE	M	NO.	A_0	D_0	V KM.
A6 to F9	<3	153	269°	+27°	21.2
	>3	133	274	35	23.4
G0 to G9	<3	224	266	32	19.9
	>3	154	279	23	40.2
K0 to K9	<4	325	282	43	18.3
	>4	112	286	31	33.1
M	<4	98	271	44	16.2
A6 to M	Giants	800	273	37	18.8
A6 to M	Dwarfs	415	281	29	31.7

The stars of each spectral type have been divided into two groups according to their absolute magnitude M, and these groups, in the case of the later types at least, can be identified with the giant and dwarf classes. We see that for the giant stars the declination D_0 of the sun's apex shows a steady increase from 27° to 44° with advancing spectral type. The same result has been found by several investigators who have studied the distribution of proper motions, and the effect seems to be very marked among the apparently faint stars studied by Dyson and Thackeray.[2] This indicates a larger proportion of stars belonging to the second stream among the K and M stars than among those of early types. The dwarf stars show a much larger value of the velocity of the sun than that deduced from the stars in general.

A very remarkable feature is the behavior of the stars of high velocity. If we group the stars according to their velocities with reference to an origin corrected for the standard solar velocity of 20.0 km. directed toward the point $A = 270°$; $D = +30°$, we find that the velocity of the sun increases remarkably with increasing speed of the stars relative to which its motion is referred.

The following table gives the result of such a computation, the stars being grouped according to velocity.

V KM./SEC.	A_0	D_0	V KM.
0 to 60.........	272°	+30°	20.6
60 to 100.........	295	+43	36.3
100 to 150.........	289	+39	76
>150.........	313	+54	209

We see from this comparison that the stars of high speed have a systematic motion which differs from that of stars of moderate speed, and that they have a tendency to move in a direction opposite to the sun's motion. The same result was indicated by Adams and Joy[3] who found from a study of the space-motions of 37 stars of high radial velocity that nearly all the apices were confined to one hemisphere.

Distribution of Velocities.—The peculiarities in the motions of the stars can best be studied by determining the frequency-function according to which the velocity-vectors are distributed. Two different types of frequency-functions have been used hitherto, namely, those based upon the two-drift theory of Kapteyn[4] and the ellipsoidal theory of Schwarzschild.[5] If use is made of three-dimensional velocity-vectors it is easy to decide between these two theories, but in order to represent the actual distribution of the velocity-vectors in all its aspects a more general type of frequency-function is necessary. In the first place the velocity-vectors were all corrected for a standard solar velocity of 20.0 km. in the direction $A_0 = 270°$ and $D_0 = +30°$. About 100 stars which had been selected for observation on account of their small proper motions were omitted since they would tend to give a "star stream nearly parallel to the direction of the sun's motion. Stars recognized as members of the Taurus Group were also omitted, as well as fainter components of double stars.

The frequency-function used is a three dimensional cosine-series with the origin in the center of the limiting planes. Such a function can be written

$$F(xyz) = a_0 + a_{100} \sin \alpha + a_{010} \sin \beta + a_{001} \sin \gamma + a_{200} \cos 2\alpha + a_{110} \sin \alpha \sin \beta + a_{101} \sin \alpha \sin \gamma + a_{020} \cos 2\beta + a_{011} \sin \beta \sin \gamma + a_{002} \cos 2\gamma + a_{300} \sin 3\alpha + a_{210} \cos 2\alpha \sin \beta + a_{201} \cos 2\alpha \sin \gamma + \dots$$

$$\alpha = \frac{\pi x}{2c_1} \qquad \beta = \frac{\pi y}{2c_2} \qquad \gamma = \frac{\pi z}{2c_3}$$

In these expressions $\pm c_1$, $\pm c_2$ and $\pm c_3$, are the six limiting planes within which the function can be regarded as an interpolation formula in three dimensions. For different values of the density F we obtain a sequence of equifrequential surfaces. We see then that α, β, and γ vary between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. The integration between certain limits as well as the determination of the coefficients in the series can be performed easily. All terms up to the sixth order have been computed, while in the case of the dwarf stars, where the limits are very large, terms up to the 10th order have been included. The direction of the x-axis is towards the intersection of the galactic plane with the equator, (in Aquila), the y axis is in the direction of galactic longitude 90°, and the z axis towards the galactic north pole.

After the coefficients in the trigonometric series had been computed, a synthesis was made for $z = 0$ and $y = 0$. Curves of equal frequency were then drawn in the xy plane and the xz plane. These curves are the intersections of equifrequential surfaces with these two planes. The total number of stars including those outside the limits have in all cases been reduced to 1000.

The general appearance of the frequency surfaces can be easily seen from the diagrams in figure 1, which give the intersections of the former with the xy and xz planes for the brighter stars of spectral types F to G, and for the dwarf stars of types G0 to M. The curves for the brighter K and M stars are similar to those for the types G_6 to K_1, although the curves are more nearly circular. The crosses (+) represent the velocities of stars belonging to the Taurus Group which were omitted in the analysis. The symbol ⊙ represents the velocity of the sun, whose velocity components in our system of coördinates are $x = +17.0$; $y = +7.4$; $z = +7.4$ km./sec.

The general conclusions to be drawn from a study of these diagrams and the series on which they are based are the following:

1. The giant stars of all types form one single "stream" with an ellip-

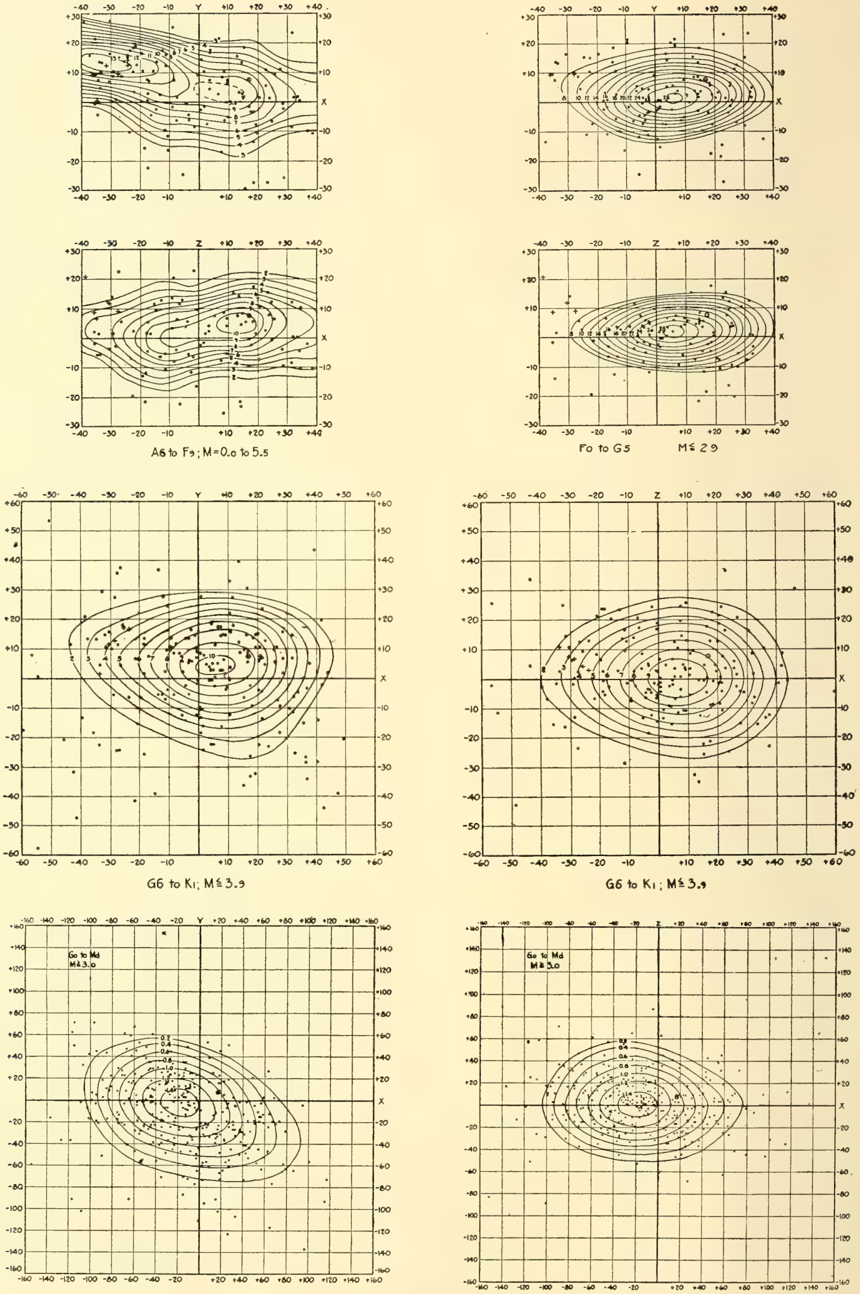

FIG. 1

Distribution of velocities of stars showing the intersections of equi-frequential surfaces of the terminal points of the velocity-vectors with the *xy* and *xz* planes.

soidal distribution; the longest and the intermediate axes in all cases lying very nearly in the galactic plane and coinciding approximately with the x and y axes. The shortest axis is perpendicular to the galactic plane. The elongation of the ellipsoids is largest for the bright F stars and decreases for later types, the distribution being nearly spherical for the M stars.

2. The most frequent velocity does not coincide with the origin but lies in all cases in the first quadrant of the xy plane, and has the approximate values $x = +5.6$; $y = +2.3$; $z = +1.6$ km./sec. Referring the sun's motion to this origin we find its velocity to be 13.8 km./sec. towards the apex $A_0 = 267°$; $D_0 = +32°$.

3. If we exclude the very brightest F stars, many of which are Cepheid variables and stars of similar spectra, and take together stars of types A_7 to F_9 of absolute magnitude 0 to 6, we find that these stars divide themselves into two streams. One of these coincides with the general group of giant stars, as can be seen from the agreement of its condensation-points with that of the giant stars of other spectral types while the other stream moves nearly parallel to the Taurus Group. This group[6] is moving towards the point $\alpha = 92°$; $\delta = +7°$ with a velocity of 44 km./sec. or in galactic coördinates with our adopted origin $x = -26$, $y = +15$, $z = +4$ km./sec. This stream comprises the majority of the stars of F type and its existence seems to indicate that a large number of these stars in all regions of the sky are traveling with a velocity nearly identical with that of the Taurus Group. Even among the brightest F stars as well as the G-type giants the presence of members of this group is indicated by the protruding form of the equifrequential surfaces in this region. This stream might be identified with Kapteyn's First Drift although the galactic latitude is not zero.

4. The dwarf stars among which have been included those with absolute magnitudes fainter than 3.0 seem to form a group of their own as regards their motions. The distribution of the velocities is ellipsoidal, the major, intermediate and shortest axes having the galactic longitudes 162°, 70° and 291° and latitudes $+8°$, $+9°$ and $+77°$, respectively. The most frequent velocity is $x = -19$; $y = -2$; $z = +1$ km./sec. which nearly coincides with Kapteyn's First Drift, while his Second Drift has been swallowed up by the ellipsoidal stream of giant stars.

There is a very marked asymmetry in the distribution of the velocities around the most frequent velocity-vector. The stars of high velocity show a tendency to avoid the first quadrant of the xy plane. This asymmetry produces a systematic difference between the most frequent velocity and the mean of the velocity-components (the centroid). If we omit the stars of most rapid motion the centroid falls very near the origin and gives us nearly the standard solar velocity. That this asymmetry is not due to the selection of stars of large proper motion moving opposite to the sun is shown by a study of the velocity-distribution of stars having very high speeds. In the diagram in figure 2 are plotted the projections in the xy plane of the velocities of stars having space motions larger than 100 km./sec. The effect of the selection of proper motions relative to the sun instead of to our adopted origin is then insignificant, and we find that whatever the spectral types or the absolute magnitude, if the velocity of the star is great, there is an avoidance of the first quadrant in the xy plane. No star of speed higher than 100 km./sec. is moving in this direction, but a large proportion of them are moving toward the third quadrant, a result in agreement with that found by Adams and Joy[3] for stars of high radial velocity.

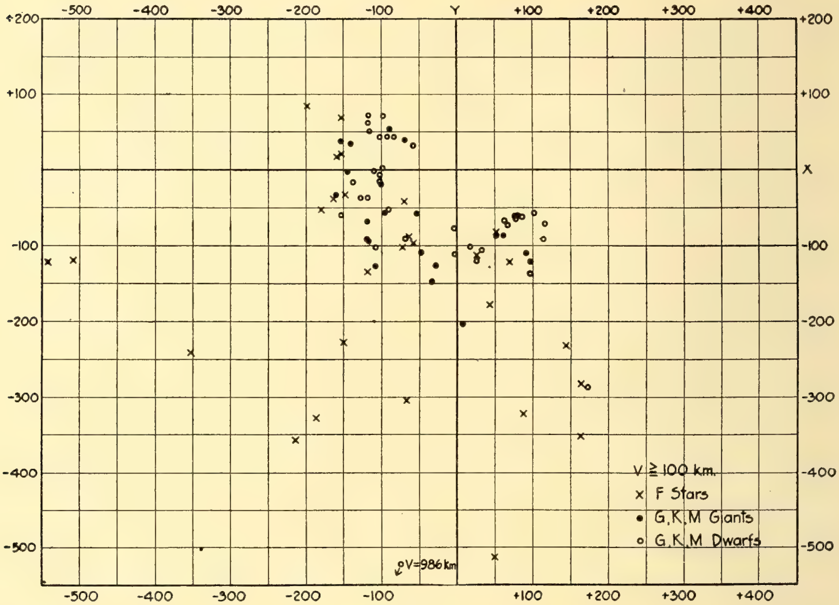

FIG. 2

Projections in the galactic plane of the velocities of stars having a speed greater than 100 km. per second.

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[1] *Mt. Wilson Contr.* No. 199, *Astroph. J., Chicago,* **53,** 1921 (13).

[2] *London, Mon. Not. R. Astron. Soc.,* **79,** 1919 (201).

[3] *Mt. Wilson Contr.* No. 163, *Astroph. J.,* **49,** 1919 (179).

[4] Kapteyn, Address before St. Louis Exposition Congress, 1904.

[5] Schwarzschild, *Göttingen Nachrichten,* 1907, p. 614. A summary of the Theories is given in Eddington's, *Stellar Movements and the Structure of the Universe,* London, 1914.

[6] Hertzsprung, *Astron. Nachrichten,* **209,** 1918.

CONTRIBUTION TO THE ENERGETICS OF EVOLUTION*

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Communicated, May 6, 1922

It has been pointed out by Boltzmann[1] that the fundamental object of contention in the life-struggle, in the evolution of the organic world, is available energy.[2] In accord with this observation is the principle[3] that, in the struggle for existence, the advantage must go to those organisms whose energy-capturing devices are most efficient[4] in directing available energy into channels favorable to the preservation of the species.

The first effect of natural selection thus operating upon competing species will be to give relative preponderance (in number or mass) to those most efficient in guiding available energy in the manner indicated. Primarily the *path* of the energy flux through the system will be affected.

But the species possessing superior energy-capturing and directing devices may accomplish something more than merely to divert to its own advantage energy for which others are competing with it. If sources are presented, capable of supplying available energy in excess of that actually being tapped by the entire system of living organisms, then an opportunity is furnished for suitably constituted organisms to enlarge the total energy flux[5] through the system. Whenever such organisms arise, natural selection will operate to preserve and increase them. The result, in this case, is not a mere diversion of the energy flux through the system of organic nature along a new path, but an increase of the total flux through that system.

Again, so long as sources exist, capable of supplying matter, of a character suitable for the compositon of living organisms, in excess of that actually embodied in the system of organic nature, so long is opportunity furnished for suitably constituted organisms to enlarge the total mass of the system of organic nature. Whenever such organisms arise, natural selection will operate to preserve and increase them, provided always that there is presented a residue of untapped available energy. The result will be to increase the total mass of the system, and, with this total mass, also the total energy flux through the system, since, other things equal, this energy flux is proportional to the mass of the system.

Where a limit, either constant or slowly changing,[6] is imposed upon the total mass available for the operation of life processes, the available energy per unit of time (available power) placed at the disposal of the organisms, for application to their life tasks and contests, may be capable of increase by increasing the rate of turnover of the organic matter through the life cycle. So, for example, under present conditions,[7] the United States produce annually a crop of primary and secondary food amounting to about 1.37×10^{14} kilogramcalories per annum, enough to support a population of about 105 million persons (equivalent to about 88 million adults) at the present rate of food consumption (4,270 kilogram-calories per adult per day). Suppose, as a simple, though rather extreme illustration, that man found means of doubling the rate of growth of crops, and of growing two crops a year instead of one. Then, without changing the average crop actually standing on the fields, the land would be capable of supporting double the present population. If this population were attained, the energy flux through the system composed of the human population and the organisms upon which it is dependent for food, would also be doubled. This result would be attained, not by doubling the mass of the system (for the matter locked up in crops, etc., at a given moment would be, on an average, unchanged) but by increasing the velocity of circulation of mass through the life cycle in the system. Once more it is evident that, whenever a group of[8] organisms arises which is so constituted as to increase the rate of circulation of matter through the system in the manner exemplified, natural selection will operate to preserve and increase such a group, provided always that there is presented a residue of untapped available energy, and, where circumstances require it, also a residue of mass suitable for the composition of living matter.

To recapitulate: In every instance considered, natural selection will so operate as to increase the total mass of the organic system, to increase the rate of circulation of matter through the system, and to increase the total energy flux through the system, so long as there is presented an unutilized residue of matter and available energy.

This may be expressed by saying that *natural selection* tends to make the energy flux through the system a maximum, so far as compatible with the constraints to which the system is subject.

It is not lawful to infer immediately that *evolution* tends thus to make this energy flux a maximum. For in evolution two kinds of influences are at work: selecting influences, and generating influences. The former select, the latter furnish the material for selection.

If the material furnished for selection is strictly limited, as in the case of a simple chemical reaction,[9] which gives rise to a finite number of products, the range of operation of the selective influences is equally limited.

In the case of organic evolution the situation is very different. We have no reason to suppose that there is any finite limit to the number of possible types of organisms. In the present state of our knowledge, or rather our ignorance, regarding the generating influences that furnish material for natural selection, for organic evolution, an element of uncertainty enters here. It appears, however, at least *a priori* probable that, among the certainly very large (if not infinite) variety of types presented for selection, sooner or later those will occur which give the opportunity for selection to operate in the direction indicated, namely so as to increase the total mass of the system, the rate of circulation of mass through the system, and the total energy flux through the system. If this condition is satisfied, the law of selection becomes also the law of evolution:

Evolution, in these circumstances, proceeds in such direction as to make the total energy flux through the system a maximum compatible with the constraints.

We have thus derived, upon a deductive basis, at least a preliminary answer to a question proposed by the writer in a previous publication.[10] It was there pointed out that the influence of man, as the most successful species in the competitive struggle, seems to have been to accelerate the circulation of matter through the life cycle, both by "enlarging the wheel," and by causing it to "spin faster." The question was raised whether, in this, man has been unconsciously fulfilling a law of nature, according to which some physical quantity in the system tends toward a maximum. This is now made to appear probable; and it is found that the physical quantity in question is of the dimensions of power, or energy per unit time, as was hinted by the writer on an earlier occasion.[11]

It may be remarked that the principle of maximum energy flux here set forth bears a certain outward resemblance to a principle enunciated by Ostwald:[12] "Of all possible energy transformations, that one takes place, which brings about the maximum transformation in a given time." This principle of Ostwald's, however, is based on entirely different grounds from those here brought forward. It is not of general applicability, and in particular, its application to systems of the kind here considered does not appear warranted.

Addendum. Since the paragraphs above were penned, the writer has received from the booksellers a copy of Professor J. Johnstone's book, "The Mechanism of Life" (1921), in which (pp. 217–221) that author touches on matters closely related to those here discussed. Professor Johnstone draws, however, a somewhat different conclusion, namely that "In living processes the increase of entropy is retarded."[13] He points out that this is true, primarily, of plants; but that among animals[14] also natural selection must work toward the weeding out of unnecessary and wasteful activities, and thus toward the conserving of free energy, or, what amounts to the same thing, toward retarding energy dissipation.

This is perhaps not wholly convincing, for the first effect of the advent of animal organisms in a world peopled with a purely vegetable population, would certainly seem to be an acceleration of the process of dissipation. It appears, therefore, that at certain stages in the evolution of the system, at the least, life must have tended to increase rather than decrease dissipation. And even if animals ultimately evolve in the direction of decreased dissipative effect, they still remain essentially a dissipative type, as compared with plants, and, to make Professor Johnstone's argument conclusive, it would seem necessary to show, not merely that the animal organism evolves in that direction, but that the system of coupled transformers, plant and animal, as a whole has so evolved.

Professor Johnstone's conclusion is, however, not essentially incompatible with the one developed in this paper. Where the supply of available energy is limited, the advantage will go to that organism which is most efficient, most economical, in applying to preservative uses such energy as it captures. Where the energy supply is capable of expansion, efficiency or economy, though still an advantage, is only one way of meeting the situation, and, so long as there remains an unutilized margin of available energy, sooner or later the battle, presumably, will be between two groups or species equally efficient, equally economical, but the one more apt than the other in tapping previously unutilized sources of available energy.

There is here a problem that will call for further investigation. In particular, it remains to be established just what is the significance of the phrase "compatible with the constraints" which, in the presentation here given, modifies the maximum principle enunciated. The present communication is intended rather a preliminary than as an attempt to say the last word on the subject. More detailed discussion is planned for another occasion.

* Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 59.

[1] *Der zweite Hauptsatz der mechanischen Wärmetheorie,* 1886 (Gerold, Vienna), p. 21; *Populäre Schriften,* No. 3, Leipsic, 1905; Nernst, *Theoretische Chemie,* 1913, p. 819; Burns and Paton, *Biophysics,* 1921, p. 8, H. F. Osborn, The Origin and Evolution of Life, 1918, p. XV.

[2] Compare also Sir Oliver Lodge, *Life and Matter,* 1906, pp. 139, 140.

[3] Lotka, A. J., *Ann. Naturphil,* 1910, pp. 67, 68; *Proc. Nat. Acad., Sci.,* 1921, pp. 194, 195.

[4] Lotka, A. J., *Proc. Washington Acad.,* **5,** 1915, pp. 360, 397.

[5] The term *energy flux* is here used to denote the available energy absorbed by and dissipated within the system per unit of time.

[6] As, for example, if the total mass of the system is capable of accretion, but only at a limited velocity, in which case the phenomenon of a moving equilibrium may present itself. Compare Lotka, A. J., *Proc. Nat. Acad. Sci.,* **7,** 1921, p. 168.

[7] Pearl, R., *The Nation's Food,* 1920, pp. 218, 203, 258, 80, 245; *Amer. J. Hygiene,* **1,** 1921, p. 598.

[8] Owing to the fact that in existing organisms the anabiotic and catabiotic functions are very largely segregated in different types (plants and animals), evolution will here operate upon systems or groups of at least two species, one species of autotrophic anabions, and one of heterotrophic catabions.

[9] Compare Lotka, A. J., *Amer. J. Sci.,* **24,** 1907, pp. 204, 216.

[10] Lotka, A. J., *Proc. Nat. Acad. Sci.,* **7,** 1921, p. 172.

[11] *Idem., Ann. Naturphil.,* 1910, p. 70. It is there suggested that the continuous energy transformations associated with the maintenance of a steady state would probably be found to play the dominant rôle, while any "latent heat" effect associated

with a change in the distribution of matter among the several species composing the system, would probably play a subordinate rôle; in contrast with the condition of affairs familiar in ordinary physico-chemical systems. This is an obvious inference from the observation that the several species of organisms are distinguished much more by structural differences than by differences in chemical composition.

[12] Ostwald, W., *Lehrbuch der allgemeinen Chemie,* 1892, vol. 2, p. 37; Siebel, J. E *Compend of Mechanical Refrigeration,* 1915, p. 88. For a discussion of the validity and limitations of Ostwald's principle see Helm G., *Die Energetik,* 1898, pp. 248; Neumann, C., *Leipziger Berichte,* 1892, p. 184.

[13] That living organisms may be capable of retarding the energy flux through the system of nature was suggested by the present writer in *Ann. Naturphil.,* 1910, p. 60.

[14] Johnstone, J., *The Mechanism of Life,* 1921, p. 220.

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*NATURAL SELECTION AS A PHYSICAL PRINCIPLE**

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Communicated May 6, 1922

In a paper presented concurrently with this, the principle of natural selection, or of the survival of the fittest (persistence of stable forms), is employed as an instrument for drawing certain conclusions regarding the energetics of a system in evolution.

Aside from such interest as attaches to the conclusions reached, the method itself of the argument presents a feature that deserves special note. The principle of natural selection reveals itself as capable of yielding information which the first and second laws of thermodynamics are not competent to furnish.

The two fundamental laws of thermodynamics are, of course, insufficient to determine the course of events in a physical system. They tell us that certain things cannot happen, but they do not tell us what does happen.

In the freedom which is thus left, certain writers have seen the opportunity for the interference of life and conciousness in the history of a physical system. So W. Ostwald[2] observes that "the organism utilizes, in manyfold ways, the freedom of choice among reaction velocities, through the influence of catalytic substances, to satisfy advantageously its energy requirements." Sir Oliver Lodge also, has drawn attention to the guidance[3] exercised by life and mind upon physical events, within the limits imposed by the requirements of available[4] energy. H. Guilleminot[5] sees the influence of life upon physical systems in the substitution of guidance by choice in place of fortuitous happenings, where Carnot's principle leaves the course of events indeterminate. As to this, it may be objected that the attribute of fortuitousness is not an objective quality of a given event. It is the expression of our subjective ignorance, our lack of complete information, or else our deliberate ignoring of some of the factors that actually do determine the course of events. Admitting, however, broadly, the directing influence of life upon the world's events, within the limits imposed by the Mayer-Joule and the Carnot-Clausius principles, it would be an error to suppose that the faculty of guidance which the established laws of thermodynamics thus leave open, is a peculiar prerogative of living organisms. If these laws do not fully define the course of events, this does not necessarily mean that this course, in nature, is actually indeterminate, and requires, or even allows, some extra-physical influence to decide happenings. It merely means that the laws, as formulated, take account of certain factors only, leaving others out of consideration; and that the data thus furnished are insufficient to yield an unambiguous answer to our enquiry regarding the course of events in a physical system. Whether life is present or not, something more than the first and second laws of thermodynamics is required to predict the course of events. And, whether life is present or not, *something* definite *does happen*, the course of events is determinate, though not in terms of the first and second laws alone. The "freedom" of which living organisms avail themselves *under the laws of thermodynamics* is not a freedom in fact, but a spurious freedom[6] arising out of an incomplete statement of the physical laws applicable to the case. The strength of Carnot's principle is also its weakness: it holds true independently of the particular mechanism or configuration of the energy transformer (engine) to which it is applied; but, for that very reason it is also incompetent to yield essential information regarding the influence of mechanism upon the course of events. In the *ideal* case of a reversible heat engine the efficiency is independent of the mechanism. *Real* phenomena are irreversible; and, in particular, trigger action,[7] which plays so important a rôle in life processes, is a typically irreversible process, the release of available energy from a "false" equilibrium. Here mechanism is all-important. To deal with problems presented in these cases requires new methods,[8] requires the introduction, into the argument, of new principles. And a principle competent to extend our systematic knowledge in this field seems to be found in the principle of natural selection, the principle of the survival of the fittest, or, to speak in terms freed from biological implications, the principle of the persistence of stable forms.

For the battle array of organic evolution is presented to our view as an assembly of armies of energy transformers—accumulators (plants), and engines (animals); armies composed of multitudes of similar units, the individual organisms. The similarity of the units invites statistical treatment, the development of a statistical mechanics of which the units shall be, not simple *material particles* in ordinary reversible collision of the type familiar in the kinetic theory, collisions in which action and reaction were equal; the units in the new statistical mechanics will be *energy transformers* subject to irreversible collisions of peculiar type—collisions in which trigger action is a dominant feature:

When the beast of prey A sights its quarry B, the latter may be said to enter the field of influence of A, and, in that sense, to collide with A. The energy that enters the eye of A in these circumstances may be insignificant, but it is enough to work the relay, to release the energy for the fatal encounter. And because evolution works with armies built up of similar units, the seemingly erratic workings of the relay mechanism (in which action and reaction are not equal, and seem subject to no simple general law) are not, in effect, erratic, but range themselves according to law and order, for those species of units, those types of transformers, are picked out for survival, whose mechanism possesses certain definite properties. Thus the principle of natural selection makes its entry into dynamics.

Further elaboration of these concepts must be reserved for a future occasion.

In systems evolving toward a true equilibrium (such as thermally and mechanically isolated systems, or the isothermal systems of physical chemistry), the first and second laws of thermodynamics suffice to determinate at any rate the end state; this is, for example, independent of the amount of any purely catalytic substance that may be present. The first and the second law here themselves function as the laws of selection and evolution, as has been recognized by Perrin[9] and others, and exemplified in some detail by the writer, for the case of a monomolecular reversible reaction.[10]

But systems receiving a steady supply of available energy (such as the earth illuminated by the sun), and evolving, not toward a true equilibrium, but (probably) toward a stationary state, the laws of thermodynamics are no longer sufficient to determine the end state; a catalyst, in general, does affect the final steady state. Here selection may operate not only among components taking part in transformations, but also upon catalysts, in particular upon auto-catalytic or auto-catakinetic constituents of the system. Such auto-catakinetic constituents are the living organisms,[11] and to them, therefore the principles here discussed, apply.

That the principle of selection is competent to yield information beyond the scope of the laws of thermodynamics has been very clearly set forth, independently, by H. Guilleminot.[12] The present writer has long realized that the principle is capable of such application; that it functions, as it were, as a third law of thermodynamics (or a fourth, if the third place be given to the Nernst principle). If he has not, before this date, explicitly stated the case, this is mainly because his writings have followed a definite, systematic plan, announced in his early publications.[13] Viewing evolution as a change in the distribution of matter among the components of a physical system, the study of evolution naturally divides itself into two fields. The one, which might be termed the *stoichiometry* of evolution, deals with mass relations: the relative amounts of the different species of matter present, and the changes in these amounts; the *kinetics* of evolution. The second field of study is the *dynamics* or *energetics* of evolution, the scope of which is sufficiently indicated by these terms. It appeared desirable to lay the foundation of the first, as the more elementary, of these fields, before proceeding to a systematic exposition of the second. This is the plan which has been closely followed by the writer in the past, and which it is hoped to develop in greater completeness in the future. Material held in reserve, and relating to the dynamics of evolution, will then be brought forward in its proper place. The present issue of this advance sheet is prompted by a recent reading of Guilleminot's book, which, through a series of mishaps, has only recently come to the writer's hand.

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*Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 60.

[1] These PROCEEDINGS, **8** p. 147.

[2] Ostwald, W., *Vorlesungen über Naturphilosophie*, 1902, p. 328.

[3] Sir Oliver Lodge, *Life and Matter*, 1906, p. 144.

[4] *Ibid.*, pp. 148, 149; *Nature*, **67**, p. 595; **68**, p. 31.

[5] Guilleminot, H., *La Matière et la Vie*, 1919, p. 121, et passim.

[6] This remark must be understood to apply only to that freedom which arises out of the incompleteness of the first and second law as determinants of the course of events. The writer does not here take sides, one way or another, in the controversies regarding free will, determinism, vitalism as distinguished from mechanistic conceptions, etc.

[7] Compare Lotka, A. J., *J. Washington Acad. Sci.*, **2**, 1912, p.7 1; Guilleminot, H., *loc. cit.*, p. 115; Johnstone, J., *The Mechanism of Life*, 1921, p. 49.

[8] Lotka, A. J., *Proc. Nat. Acad. Sci.*, **7**, 1921, pp. 194, 196.

[9] Perrin, J., *Traité de Chimie*, 1903, vol. 1, pp. 142–143; Chowlson, O. D., *Lehrbuch der Physik.*, 1905, vol. 3, p. 499; *Scientia*, **3**, 1910, p. 51; Lotka, A. J., *Science Progress*, **14**, 1920, p. 406.

[10] For recent substantiation of some of the details of the presentation there given see Baly, E. C. C., *Nature*, vol. 109, 1922, p. 344.

[11] D'Arcy Thompson (*Growth and Form*, 1917, p. 132) attributes the origination of this concept to Chodat, quoted by Monnier, A. (*Publ. Inst. Bot. Univ. Genève*, (7) **III**, 1905. There seem to be, however, some earlier indications of the same thought. The following bibliography, which makes no pretense of completeness, is culled from works ready at hand: Errera, L., *Revue de l' Université de Bruxelles*, **5**, 1899–1900. May issue; Ostwald, W., *Vorlesungen über Naturphilosophie*, 1902, pp. 342, et seq.; Bastian, H. C., *The Nature and Origin of Living Matter*, 1905, p. 46, et seq.; Robertson, T. B., *Arch. Entwickelungsmechanik Org.*, **25**, 1908, p. 581, **25**, p. 108; Ostwald, Wo., *Die zeitlichen Eigenschaften der Entwickelungsvorgänge*, 1908; Hatai, S., *Anat. Rec.*, **5**, 1911, p. 373; Enriques, *Biol. Centralbl.*, 1909, p. 337; Lotka, A. J., *Z. physik. Chem.*, **72**, 1910, p. 511; **80**, 1912, p. 159; *J. Phys. Chem.*, **14**, 1910, p. 274; *Proc. Nat. Acad. Sci.*, **6**, 1920, p. 275; Pearl, R., *Amer. J. Hygiene*, **1**, 1921, p. 592.

[12] Guilleminot, H., *loc. cit.*, p. 118, 154, et passim.

[13] Lotka, A. J., *Am. J. Sci.*, **24**, 1907, p. 216; *Ann. Naturphil.*, 1910, p. 74.

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PROCEEDINGS
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Volume 8 JULY 15, 1922 Number 7

THE PHYSICAL EXAMINATION OF HEARING AND BINAURAL AIDS FOR THE DEAF

By R. L. Wegel

Research Laboratories of the American Telephone and Telegraph Company and the Western Electric Company, Inc.

Read before the Academy, April 25, 1922

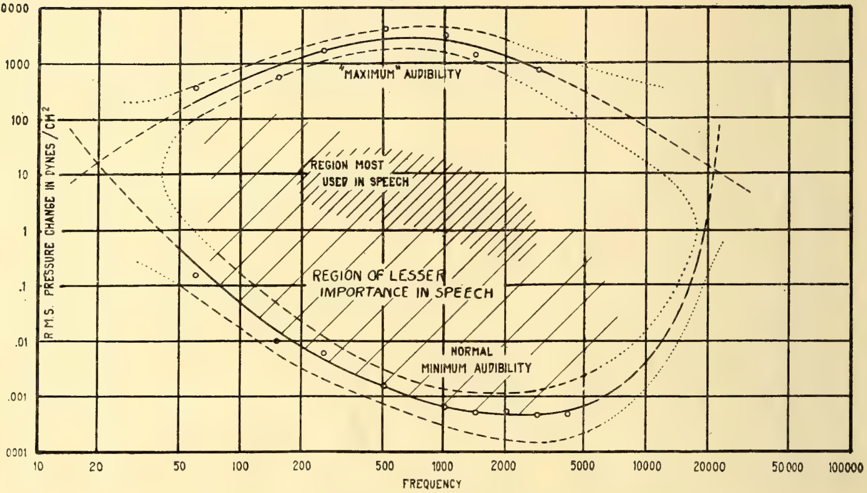
1. *Introduction.*—The function of the auditory sense is to detect sounds of various kinds and wave shapes varying over a range of pressure on the ear drum of from about .001 dyne to 1,000 dynes and over a considerable part of this range to differentiate with certainty between complex sounds so nearly alike that no existing physical apparatus can separate them. The binaural feature adds a sense of orientation with respect to a source and uniform sensitivity for sounds approaching from different directions. The abnormal auditory sense may be regarded as lacking more or less in (*a*) range of sensation (frequency and intensity); (*b*) quality of sensation in various regions of the range; (*c*) the binaural sense. Apparatus and methods have been developed by means of which the outstanding elements of these functions can be measured and to a limited extent compensated for.

In a sense this paper prefaces a more detailed discussion of the audiometrical and pathological phases of abnormal hearing which will be presented next week by Dr. E. P. Fowler and the author before meetings of the American Otological Society and the American Rhinological, Laryngological, and Otological Society.

2. *Minimum Audibility.*—Figure 1 shows a plot of the minimum audible pressure on the average of 72 normal ears taken throughout a range of frequency of from 60 to 4,000 cycles. Both the intensity and frequency scales are logarithmic. This curve has already been published (The Frequency Sensitivity of Normal Ears, by H. Fletcher and R. L. Wegel, these Proceedings, January, 1922, and in the *Physical Review*). Although all skew errors in the determination of the average curve have not been eliminated, an investigation has shown that they are so small as not to effect the utility of the curve for the purpose of measuring deafness. Among

the errors which obviously tend to raise this curve might be mentioned—lack of attention, low mentality of the observer, noise in the observing room, and abnormality of hearing. Care was taken to reduce all these errors to a minimum without actually making separate quantitative measurements of them on a rigorous statistical basis.

Fig. 1



AUDITORY SENSATION AREAS

The statistical deviation from the mean varies irregularly with frequency very likely due mostly to the external anatomical variations which cause deviations in the dynamical constants of the transmission system from sound source to the ear drum. The dotted lines following the curve of minimum audibility represent approximately the "standard deviation."

3. *Maximum Audibility.*—The curve marked "Maximum audibility" represents the pressure on the average of 48 normal ears required to produce the sensation of feeling. A sound much louder than this is painful. The measurements were taken through a range of from 60 to 3,000 cycles. The standard deviation lines are also given from which it will be seen that this curve is quite as definite as that of minimum audibility. While this point of feeling probably has no relation to the auditory sense it does serve as a practical limit to the range of auditory sensation. A few observations indicate that people with abnormal ears have a point of feeling sound which is not greatly different from that of normal ears, but this of course depends on the type of abnormality. The intensity for feeling is about equal to that required to excite the tactile nerves in the finger tips.

4. *Lower and Upper Frequency Limits of Hearing.*—The curves of mini-

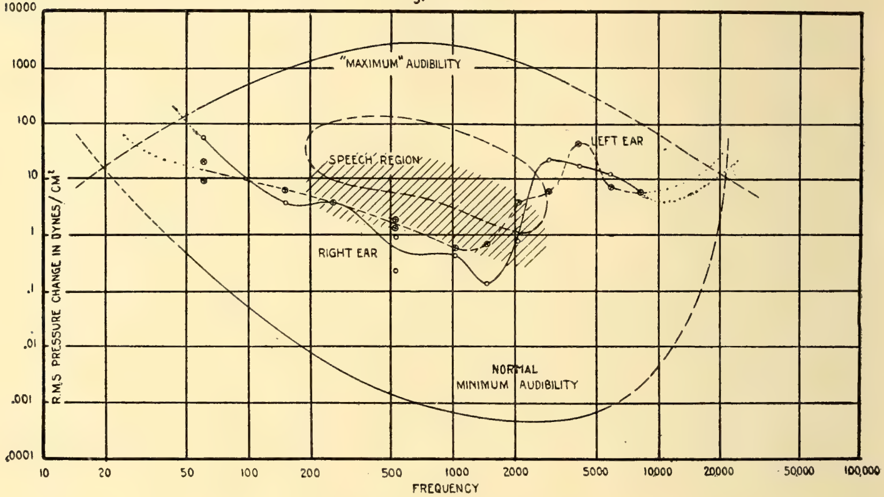
mum audibility in figure 1 will be seen to have been extrapolated to the points of intersection at high and low frequencies. The feeling sensation in the middle range of frequency is first a tickling sensation and then becomes acutely painful as the loudness is increased. As the frequency is decreased the sensation of feeling becomes milder until at frequencies around 60 cycles it is sensible as a flutter, but still quite different from the sense of audition. As the frequency is still further decreased to a point where the hearing and feeling lines appear to intersect, it is difficult to distinguish between the sense of hearing and that of feeling. The low point of intersection of two normal curves of minimum audibility and feeling sense may therefore be taken arbitrarily as the lower tone limit of audibility. For frequencies lower than this it is easier to feel than to hear the air vibration. For any one individual this point cannot be determined with accuracy on account of the variation in judgment, but can be determined by extrapolation. A similar intersection of the two curves occurs at some very high frequency. The extrapolation upward of the curve of minimum audibility is consistent with some recent observations of Mr. C. E. Lane of the University of Iowa. This work will be published shortly in the *Physical Review.*

This gives a rational way of defining the two frequency limits of audibility and suggest a method of making an accurate determination of them. Measurements of these limits which have been made in the past are questionable because the intensity factor has been neglected. At the lower limit of audibility the excursions of the diaphragm and ossicles of the middle ear are probably so large that the nerves feeding these movable parts are stimulated and therefore the measurements at very low frequencies probably give data related to the pathological condition of the apparatus of the middle ear and this independently of the stimulation of the auditory nerve. This point is probably related to the tests on flexibility of the ear drum due to the application of air pressure as observed by otologists in examinations. Otologists usually associate loss of sensitivity at low frequencies with obstructive deafness if there is no loss at high frequencies.

5. *Sensation Area.*—From the combined standpoint of utility and logic the logarithmic relation between stimulus (pressure variation) and sensation can be assumed. The elliptical area between the two curves may then be taken to represent an area of sensation which is characteristic of the normal ear. Any point within this area represents a definite auditory sensation in frequency and intensity.

An abnormal ear may be regarded as having an area of sensation which is smaller than the normal area but included within it. Figure 2 is a plot of the minimum audibility of the right and left ears for a man (CHK) suffering from a catarrhal deafness. The area between this curve of minimum audibility and the curve of feeling is his area of sensation. The area of sensation which is most utilized in the interpretation of speech is represented approximately by the shaded area in figures 1 and 2. (See "Analysis of the Energy Distribution in Speech," I. B. Crandall and D. MacKenzie, *Phys. Rev.*, Mar. 1922.) It will be seen that CHK retains about 50 or 60% of the normal amount of sensation. He hears and interprets conversation with some difficulty. The CHK curves pass through the speech area. In order to make him hear well, the speech area must be raised to a higher level of intensity or loudness as indicated by the dotted curve.

Fig. 2



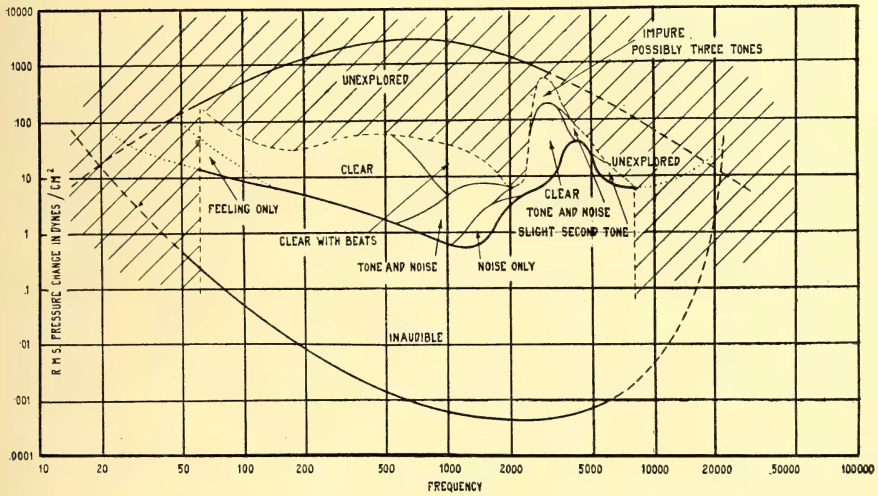
SENSATION AREAS (CHK)

6. *Importance of Various Intensities in Speech.*—It is interesting to speculate on how CHK interprets speech. It has been shown (H. Fletcher at Franklin Institute meeting Mar. 30, 1922) that the intensity of speech may be varied over perhaps 70–80% of the range of sensation without serious loss of intelligibility to the normal ear. As the sound intensity is decreased, the intelligibility drops very suddenly to zero at minimum audibility. A similar drop is to be expected at an intensity so loud as to be painful. It is evident therefore, that the range throughout which speech is intelligible for CHK is very considerably limited as compared to normal. It is possible to design a deaf set which increases the intensity of the principal speech region to any desired place within the abnormal sensation area and so in a measure compensate for this narrowed range.

The region in figure 2 marked "Region of Lesser Importance in Conversation" corresponds to stimuli in conversation of lesser energy content such as the minor shadings and fainter consonant sounds. While it is physically possible to produce an amplification of speech so that this region is raised into the diminished area it is impracticable to do so because of the pain which would be caused by the louder components. A diminution in sensation area can therefore be only partially compensated for. In case the area is extremely narrow a deaf set furnishing optimum volume can only serve as an aid to lip reading.

7. *Quality of Hearing.*—The sensation of a normal ear at any point in the auditory sense range (figure 1) may be described by any number of different adjectives, such as for example, "clear," "musical," "even," "sustained," "smooth," "pure," etc. Such a description may in fact be taken as a reasonable indication that the quality of sensation at the point in question is normal. Abnormal ears sometimes experience a subjective degeneration of quality of pure stimuli which they describe as "rough," "harsh," "sharp," "buzzing," "vibrating," "hissing," etc. These sounds are independent of any tinnitus or head noises which the patient may have. Figure 3 shows various regions of the sensation area which are degenerated in the case of CHK. The shaded area was not explored. The boundaries of the degenerated regions are usually more sharply marked than those of minimum audibility and the sensations in these areas are so radically different from the sensation of a pure tone that it is with difficulty that the patient is convinced that the stimulation is the same pure tone to which he has been listening at the other intensities. The subject of these tests is a violinist and capable of better descriptions and finer distinctions than average.

Fig. 3



QUALITY AREAS-LEFT EAR (C.H.K)

Since all speech sounds may be considered as stimuli composted of various frequency components of certain intensities the sensation caused by such a sound may be represented on this plot by points, or by a line running provided the sound has a band spectrum. If the points or line falls within the sensation area the sound is audible. It is easy to see that if the points or the part of the line which represent those frequency components most essential to interpretation of the sound, fall within any of these abnormal areas, the sound is very likely to be misinterpreted. This adds a further source of loss in articulation to that already observed due to a narrowing of the sensation range.

Many practically normal ears have very small abnormal areas. They have always been found near minimum audibility and therefore have little influence on the hearing of the individual. They seem to be associated with catarrhal conditions although this cannot be stated positively.

8. *Binaural Sense.*—The normal individual has learned to interpret the differential sensations of the two ears to an advantage. It helps him to locate the direction from which sounds come, to have a sort of sense of orientation with respect to sounds approaching from different directions, and whether for physical or for purely psychological reasons to assist in focusing of the attention on one sound of a large number. Two ears also assist the individual in perceiving equally well sounds coming from different directions. When one ear becomes insensitive even though the loss is small, the use of the binaural sense disappears and after a time is not missed. For the binaural sense to be most effectively utilized it is necessary that the ears be very nearly alike. In many cases of deafness one ear is much more deaf than the other and the subject does not utilize the binaural sense. When a binaural deaf set is made and fitted to a person with compensating sensitivity for the two ears so that both hear the sounds equally loud, the sensation is usually so novel, that if the patient is actually able to experience a binaural sensation he is very much pleased. Usually, however, he has not used his binaural sense for so long a time that it takes a considerable amount of practice before he is able to have binaural experiences. It may be noted in this connection that the same experience is encountered in fitting the eyes with glasses. It is found that people with two eyes which are slightly different do not see stereoscopically but if glasses are made so as to compensate and make the eyes nearly alike it usually takes a certain time of practice before the sense of perspective can be brought back.

An audiometer similar to the one used in the measurements recorded and a binaural deaf set have been set up here for inspection.

THE PRODUCTION OF AN E.M.F. ON CLOSED CIRCUIT BY A LIGHT EFFECT ON ARGENTITE

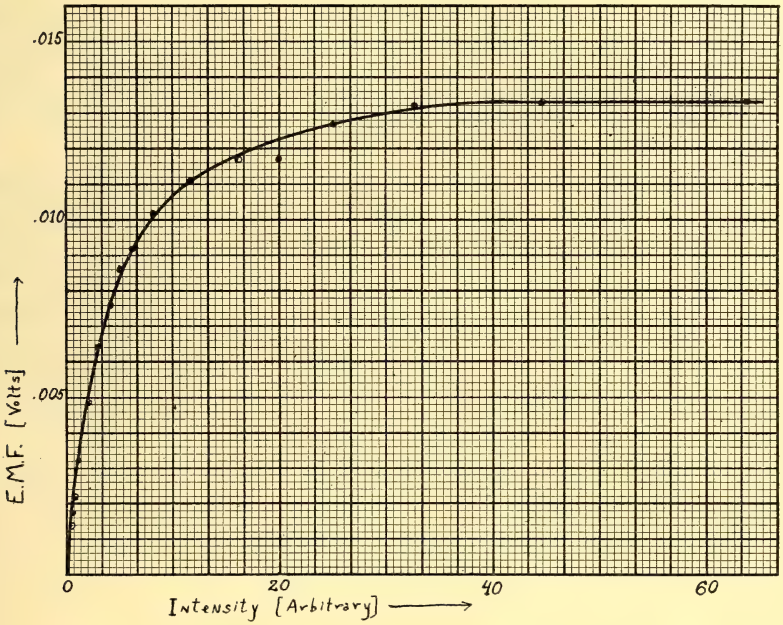
By H. Horton Sheldon and Paul H. Geiger

Physical Laboratory, University of Michigan

Communicated, May 12, 1922

While obtaining data on the change of resistance of Argentite (Ag_2S) with change of intensity of illumination, it was observed that under proper conditions a deflection of the galvanometer was secured on illumination when no external E.M.F. was applied to the circuit. The illuminated contact was always the negative end of the crystal.

To study this effect further, the crystal was connected directly across the galvanometer and all factors eliminated which might in any way have an effect on the galvanometer. It was not difficult to identify the phenomenon as distinct from pressure effects, etc., but it required more careful investigation to make certain that it was not a thermoelectric effect due to the heat radiated from the lamp. The fact that the crystal turned out to be but little affected by the longer wave-lengths somewhat simplified this procedure. A heating coil radiating much more heat than the lamp by a thermopile test, was put in place of the lamp, but the result produced was extremely slight in comparison, even when the contacts were blackened by smoke from naphthalene.



The behavior of the galvanometer was also an indication of the distinction. In the case of the heater the galvanometer showed a steady creep to the maximum value, interpreted as due to the gradual heating of the exposed contact. In the case of the light the maximum was reached in approximately 13 sec., which was the quarter period of the galvanometer. A piece of plate glass an inch thick reduced the effect of light about 30%.

Having satisfied ourselves then, that the major portion of the effect was in the visible region a series of experiments were carried on with varying intensity in an attempt to find an analogy with the photo-electric effect. For this purpose a 16 c.p. bulb, such as used in an automobile head light, was employed, being chosen because of approximating closely to a point source. The intensity was varied by sliding the lamp along an optical bench, and not by any change in energy input which might vary the wave-lengths in any way. The E.M.F.'s set up by the light action were balanced out by a potentiometer method, this null method giving very consistent results. Figure 1 shows E.M.F. in volts against intensity on an arbitrary scale.

It would seem from this that it is not similar to what is commonly called the photoelectric effect, but while this is probably true, there are other things which might alter the shape of the curve.

First, the shape of the curve might depend on whether we start our readings from high or from low intensity, since resistance has been found to increase with the length of time the current flowed. This was not the case here, however, for the same curve was obtained from either direction, since a little practice will enable one to succeed in balancing out the effect very quickly.

Secondly, the resistance has been found to change with intensity of illumination. Such a decrease in resistance would cause a lower E.M.F. across the crystal in proportion to the rest of the circuit, so that the values of E.M.F. if corrected, would be raised as we go out along the curve. Since, however, the crystal resistance may be from 200,000 to over 1,000,000 ohms, and the galvanometer resistance was only 400 ohms, the effect would be small and would not materially change the shape of the curve.

It would seem then, from present data, that this is not a true photoelectric effect as the term is now used, but probably a light effect on the contact potential.

If we look again at the figure, there is to be seen a point which would suggest a bend which might indicate a secondary effect, however the accuracy of our readings do not permit of a certainty in this respect as yet. This is being investigated as well as the variation with wave-length and it is hoped that these results may be published in the near future.

The galvanometer used is the Leeds Northrup high sensitivity type having a resistance of 400 ohms and a sensitivity of 6.02×10^{-11} amps./mm. as now used. In this work it has been critically damped. The deflections for illuminations such as used ran as high as 100 cm. The contacts used were very heavy strips of copper, about 2″ by 10″ and 1/8″ thick; they were blackened by smoke from burning naphthalene except at the point of contact, and were pressed firmly against the crystal by weights placed on them. The crystals were natural and several specimens were used, obtained from the following localities: Frieberg, Saxony, Germany; Schneeberg, Saxony, Germany; Batopilas, Mexico; and Arispe Sonora, Mexico.

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*POSITIVE AND NEGATIVE PINHOLE RESONATORS**

By Carl Barus

Department of Physics, Brown University

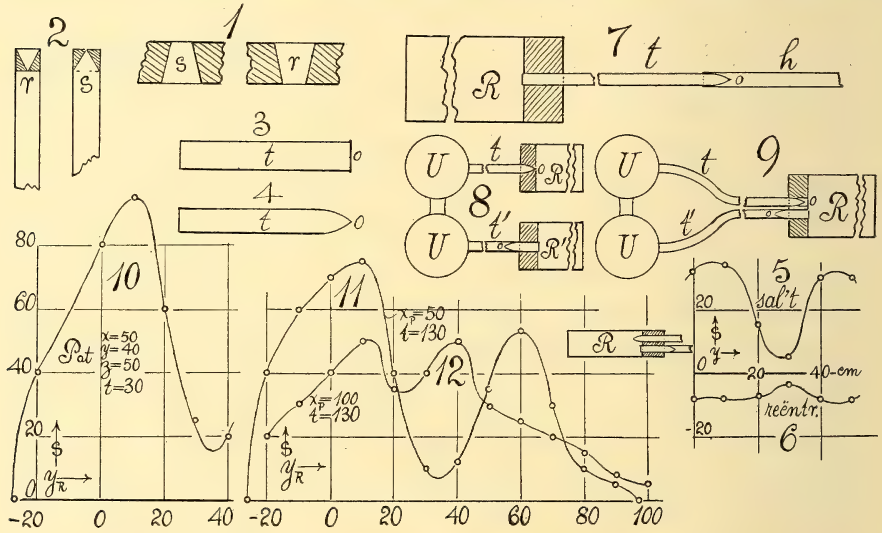
Communicated, May 11, 1922

Improvement of the Pinhole Resonator.—The plan immediately suggesting itself was the trial of an adjustable needle valve pinhole. The apparatus was made of 1/4 inch brass tube, with a conical point carrying the pinhole at its end. The tube was closed by a long nut in which a waxed screw terminated by the needle was movable. The pinhole could thus be completely closed, or opened in any degree by approaching or withdrawing the screw. The new pinhole was inserted into the tuned resonator, as usual. After many trials, nothing was obtained with this promising apparatus, until the needle was completely withdrawn. In fact the best pin holes obtained for the resonator, were made from glass quill tubes drawn to a blunt closed cone. It is necessary to open and enlarge the hole by grinding; or to close the conical end by gentle fusion and regrinding when cold.

To throw further light on this intricate subject, it seemed advisable to operate with pinholes in soft sealing wax, as these could be more easily shaped. In this way much information was obtained. A thin sheet of wax was spread on thin-paper closing the end of the quill tube. When the wax was punctured by a needle from the inside of the tube, producing a conical hole as exaggerated at *S* in figure 1, the fringe deflections were positive and reached a maximum with the right size of hole. But when the thin wax sheet was punctured from the outside of the quill tube (suggested at *r* fig. 1.), the deflection obtained was to the same degree negative. In other words, in passing from the salient pinhole to the reëntrant pinhole, acoustic pressures pass to acoustic dilatations; or the pressure excess is on the reëntrant or concave side of the conical pinhole. It follows therefore that a cylindrical pinhole must be inactive, if used with a resonator.

The remarkable efficiency of glass quill tube probes, with the pin hole at a conical end, ground off, thus finds its explanation; for these are in conformity with *s*, figure 1.

To further verify the new results, holes were bored with a fluted conical reamer in small discs of brass and these were then soldered to the ends of quarter inch-brass tubes, as shown at *r* and *s* in figure 2. The cones ended in pinholes, drilled from within and without, respectively, by fine needles. The case *s* gave positive deflections, *r*, negative deflections. It was found that on enlarging *r* from within with a needle, the deflections became positive; on further enlarging the pinhole, now from without, it became negative again. This proves, conformable with the earlier evidence, that only a very small depth of pinhole (probably of the thickness of a piece of paper) is effective and the remainder of the cone without importance.



Fine slits cut in wax behaved similarly to the pinholes above mentioned, producing pressure for a salient wedge, and the reverse. The fringe deflections, *s*, of figure 5, in which a resonator with a salient pinhole moves in *y* when tested with a reëntrant pinhole resonator, supplied the results shown in figure 6. The maxima and minima are throughout negative. Both salient and reëntrant pinholes function in the same way, with an inversion of sign, but without change of sign for the same pinhole.

The question is thus adduced, as to whether the pinhole resonator with a reversed cone, figure 7, will also produce a reversal of fringe deflections. On trial an affirmative answer was obtained at once. As a rule the negative deflections are numerically smaller than the positive deflections. It was further found that the negative deflection decreases with the length of connector tube *t*, from the bottom of the resonator *R* to the reëntrant pinhole O, figure 7, the pipe h beyond being the rubber connection to the U-gauge. The negative pinhole resonator runs down in sensitiveness much more rapidly (if treated in this way, pinhole remote), than if the pinhole is fixed in the bottom of the resonator, and a connector h of any length, communicates with the U-gauge.

Positive and Negative Pinhole Resonators Coöperating.—It is obvious that the positive (pinhole cone O salient) and negative (pinhole cone O' reentrant) pinhole resonators, R and R', figure 8, must add their respective pressure effects at the gauge U U', if they are provided with independent connector tubes t and t'. A large number of such experiments all indicated that the separate deflections may be added, at least very nearly. The sensitiveness may thus be nearly doubled. It is probable moreover that two resonators will interfere less with each other than two positive or two negative resonators (i.e., two identical resonators, generally), if placed a short distance apart. The deflection in fact, falls off slowly and does not quite vanish even with a mere crevice between the mouths. If the two resonators are parallel, side by side and mouth near mouth, the conditions are even better. Hence if R R', figure 8, are placed side by side, and the connector tubes t, t' are of pure rubber, elongated if necessary, the twin apparatus is available for exploration.

With short connectors t t' in the interest of greater sensitiveness, I made a number of tests to find the maximum distance for which the twin resonators (fig. 8) would respond. They were able to hear an organ pipe to a distance of 6 meters, the deflection being then about a fringe.

Although a single resonator with a branched tube is ineffective, the suggestion of using the bottom of the resonator R figure 9, as a branch point for the connector tubes t t' to the gauge U U', seems highly promising. One and the same resonator here functions in both (positive and negative) capacities. The design succeeded with, as it seemed, but little sacrifice of sensitivity. Figure 10 shows the fringe deflections S, obtained along y, for the case of short connector tubes t t' (30 cm. long), with the pipe (at $x = 50$, $y = 40$, $z = 50$ cm.), near the center of the table. In figure 11 the same result is extended with long connector tubes ($t = 130$ cm.). In curve 12 finally, the results refer to a more distant pipe at $x = 100$ cm., $y = 40$, $z = 50$ cm. All curves fall off quite abruptly at the two edges of the table $y = -27$ and $+93$ cm. The extreme importance of tuning was noteworthy throughout; mild notes frequently giving large deflections, sometimes out of field, whereas strong notes of the same pitch to the ear dropped the fringes back to zero. A variety of experiments were made with this ± resonator with results like the above which need not therefore be reported.

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* Advance note from a Report to the Carnegie Institution of Washington, D. C.

THE EVALUATION OF QUANTUM INTEGRALS

By Paul S. Epstein

Norman Bridge Laboratory of Physics, Pasadena, California

Communicated, May 24, 1922

Rather a long time after its publication, my attention was drawn to a paper by Mr. E. C. Kemble[1] dealing with the evaluation of the integral

$$J=2\int_a^b\sqrt{f(q)}\,dq, \tag{1}$$

where a and b denote the roots of the radicand. Before preparing his own method of computation, Mr. Kemble criticizes two procedures used by other authors and rejects them as inadequate and even as erroneous. This judgment cannot remain uncontradicted for in reality these procedures are both perfectly correct and very efficient. Since the integral under consideration was first introduced into the theory of quanta by the writer of this note[2] in order to formulate his quantum conditions for conditionally periodic motions, the moral obligation of putting things right is his.

The first method to which Mr. Kemble objects is used when $f(q)$ can be expressed in the form

$$f(q) = \varphi(q) + \alpha\psi(q) \tag{2}$$

where $\varphi(q)$ is quadratic in q, α denotes a constant and $\alpha\psi(q)$ is small compared with $\varphi(q)$. The most obvious method consists in developing $\sqrt{f(q)} = Q(\alpha)$ in powers of α

$$Q(\alpha) = Q(o) + \alpha Q'(o) + \frac{\alpha^2}{1.2}Q''(o)+\dots \tag{3}$$

and integrating this series termwise. Mr. Kemble remarks that the integrals of the higher terms cannot be calculated, "because the higher derivatives of $\sqrt{f(q)}$ with respect to α become infinite at $q=a$ and $q=b$." It seems to me that the latter statement is not sufficient to draw such a conclusion; in the coefficients of series (3) there is, indeed, substituted o for α, so that they remain finite at the values $q=a$ and $q=b$, and so that the case requires a further investigation. That the roots of the function $\varphi(q)$ can also in a certain sense be regarded as limits of integration can be deduced only if the method of complex integration is used, but the latter supplies at the same time the means for overcoming the difficulty of the integrand becoming infinite at those points. This method was given by Riemann[3] just for the treatment of integrals of the type (1) and was first applied by Sommerfeld[4] to problems of the theory of quanta. It is not necessary to expand upon it for it is explained with much detail in the books both of Riemann and of Sommerfeld.[5] Therefore the assertion of Mr. Kemble that the expansion (3) is applicable only when terms of the second and higher order in α are negligible is entirely unfounded. An expansion of this type was first introduced into the theory of quanta by the present writer in his work on the Stark effect and, as a matter of fact, the computation was carried through by him not only for the term of the first order, but also for the term of the second order[6] and proved to be as easy as can be desired.

The method followed by F. Tank in a paper referred to by Mr. Kemble is but a slight modification of this procedure. The purpose of Tank is to find an expansion for the integrand $\sqrt{f(q)}$ when it is not given in the form (2). This purpose is reached by developing $f(q)$ in a power series from the point $q=d$, which makes $f(q)$ a maximum. If $q-d=\xi$ and if H denotes the maximum value of $f(q)$, this function can be represented in the form

$$f(q) = H - \alpha\xi^2 - (\beta\xi^3 + \gamma\xi^4 + \dots) = H - \alpha\xi^2 - \Delta,$$

wherefrom

$$\sqrt{f(q)} = \sqrt{H-\alpha\xi^2} - \frac{1}{2}\,\frac{\Delta}{\sqrt{H-\alpha\xi^2}} - \frac{1}{8}\,\frac{\Delta^2}{(\sqrt{H-\alpha\xi^2})^3}, - \dots$$

To this series the method of complex integration can be applied termwise in exactly the same way as to expansion (3). Since the complex path of integration amounts to a real one between the roots of the radicand $H-\alpha\xi^2$, whenever this real integral is convergent. Tank was perfectly justified in taking as limits $-(H/\alpha)^{1/2}$ and $+(H/\alpha)^{1/2}$. Moreover I cannot agree with the opinion of Mr. Kemble that "the expansion is not usually convergent throughout the interval of integration." It can be shown in a general way that the problem of convergence cannot lead to any difficulties.

Summarizing, we can say that both methods in question are correct and have been successfully applied to the treatment of important problems. The new procedure proposed by Mr. Kemble may answer its purpose as a method of computation, but it surely does not equal the old ones in directness and mathematical elegance.

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[1] E. C. Kemble, *Proc. Nat. Acad. Sci.*, **7,** 1921 (283).

[2] Paul S. Epstein, *Ann. Physik.*, **50,** 1916 (489).

[3] B. Riemann, *Schwere, Elektrizität, Magnetismus,* section 27, Hannover, 1880.

[4] A. Sommerfeld, *Physik. Zs.*, **17,** 1916 (500).

[5] A. Sommerfeld, *Atombau u. Spektrallinien,* pp. 476–482. Braunschweig, 1921.

[6] P. S. Epstein, *Ann. Physik.*, **51,** 1916 (184).

INVISIBLE SUN-SPOTS

By George E. Hale

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Communicated, May 26, 1922

The vortex hypothesis, proposed in 1908, assumes that a sun-spot resembles a vast tornado in which electrified particles, due to ionization in the solar atmosphere, are rapidly whirled. The invariable presence of a magnetic field, caused by the revolving charges, confirms this view,[1] which is also supported by various other results of observation with the spectrograph and spectroheliograph. Subsequently it was found that most sun-spots are associated in pairs, of opposite magnetic polarity, in which the preceding (western) spot is usually the larger of the two. 61 per cent of 970 spots examined in the years 1915–1917 were of the bipolar type, while 33 per cent were unipolar. All but 11 per cent of the unipolar spots, however, showed a tendency toward the bipolar type, indicated by trains of calcium flocculi following (less often preceding) the single member.[2] Frequently such groups oscillate between the unipolar and bipolar types, one or more small spots appearing or disappearing within the mass of calcium flocculi. This peculiarity has led to a search for invisible spots, regarded as vortices giving appreciable magnetic fields, in which the cooling due to expansion is insufficient to cause perceptible darkening of the sun's surface.

Preliminary attempts to increase the contrast by the exclusive use of ultra-violet light did not prove successful, and a second method of detecting the slight difference of radiation by a heat-measuring instrument (bolometer, thermopile, or radiometer) or a photo-electric cell has not yet been tried. A third method, however, has given very satisfactory results. This consists of a simple device for rendering weak magnetic fields visible by their Zeeman effect.

The iron line λ 6173, observed in the second order of the 75-foot spectrograph of the 150-foot tower telescope, appears as a wide triplet in the strong magnetic fields of large sun-spots. In very small spots, where the field is weak, it is merely widened, but either edge can be cut off by a Nicol prism and quarter-wave plate mounted above the slit of the spectrograph. When searching for invisible spots a superposed half-wave plate, oscillated back and forth by a small electric motor, is used to reverse the action of the quarter-wave plate when it comes before the slit. While the motor is running, a promising region on the sun (marked by faculae or calcium flocculi) is caused to pass slowly across the slit, and the λ 6173 line is carefully watched. The presence of an invisible spot is betrayed by the apparent oscillation of the corresponding part of the line, due to the alternate extinction of its red and violet edges. In this way magnetic fields of an intensity of 200 gausses can be detected.

A systematic search for invisible spots was begun by Mr. Ellerman and the writer on November 19, 1921, when two were found, with negative magnetic fields of 400 and 300 gausses, respectively, in the calcium flocculi following a positive single spot, No. 1920. One of these may not have belonged to this spot group, as it was about 4° south. Both were again detected by Mr. Ellerman on the following day, with slightly weaker fields. On November 21 only one could be found, and its position was now indicated by a faint marking, which on the following day became a visible spot, with a field-strength of 300 gausses. The other invisible spot to the south was not found, but another spot, preceding it and not previously detected magnetically, had also appeared.

Several other invisible spots have since been observed. The two small spots constituting the following member of No. 1919, which disappeared to the eye on November 25, were still detected magnetically by Nicholson on November 26 and on November 27, with positive fields of 300 and 200 gausses respectively. No. 1924, recorded from visual observations as a positive unipolar spot, was found by Ellerman on December 12 to be followed by a negative magnetic field of 200 gausses; both this and another invisible spot of negative polarity, observed by him on the following day, appeared as visible spots on December 14. On January 9, 1922 Ellerman observed a positive invisible spot (300 gausses) following the eastern negative spot of a regular bipolar group, No. 1932. No spot was observed in this position either before or after January 9. A small positive spot which appeared in group No. 1938 on January 23, 1922, though not observed either visually or magnetically on the two following days, was detected by Ellerman as a positive invisible spot (400 gausses) on January 26. Following the unipolar spot No. 1942 an invisible negative spot (300 gausses) was observed on February 6 by Ellerman and on February 26 he found a positive invisible spot (500 gausses) following No. 1945 (negative) and a negative one (300 gausses) following No. 1947 (positive). No visible spots were seen in either of these positions. On March 15 Nicholson detected a negative invisible spot (300 gausses) following No. 1953 (positive) in the region where the following spots had disappeared.

In an examination of our earlier magnetic records we have found nine other cases in which a local magnetic field was observed where no spot was recorded. Most of these were probably invisible spots, but as very small visible spots at the points in question might have escaped notice, and as the magnetic observations were not followed up, they are not included here.

The systematic observation of invisible spots, especially during the periods preceding and following the visible life of those that reach maturity, should assist materially in revealing the cause of spot formation. Their frequency of occurrence is probably much greater than these first results suggest, as our winter observations have been seriously limited by exceptionally cloudy weather, poor seeing, and low solar activity.

[1] Hale, On the Probable existence of Magnetic Fields in Sun-spots. *Mount Wilson Contr.*, No. 30; *Astroph. J., Chicago,* **28,** 1908 (315–343).

[2] Hale, Ellerman, Nicholson, and Joy. The Magnetic Polarity of Sun-spots. *Mount Wilson Contr.*, No. 165; *Astroph. J., Chicago,* **49,** 1919 (153–178).

COMPARISON OF THEORY WITH OBSERVATION FOR THE MINOR PLANETS 10 HYGIEA AND 175 ANDROMACHE WITH RESPECT TO PERTURBATIONS BY JUPITER

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Read before the Academy, April 24, 1922

The perturbations of the minor planets 10 Hygiea and 175 Andromache, developed by Dr. Estelle A. Glancy and Dr. Sophia H. Levy on the basis of the revision of von Zeipel's formulae and tables of minor planets which have a mean motion approximately twice that of Jupiter, (Hecuba Group), have lately been severely tested by comparison of recent observations with the computed places, for which the numerical work was performed by Dr. H. Thiele. This revision of von Zeipel's theory will soon appear as the third memoir of Volume 14 of the MEMOIRS OF THE ACADEMY. The results of these comparisons are highly encouraging and prove that the revised tables for the Hecuba Group more than meet the practical requirements of a satisfactory representation of the motion of the minor planets belonging to this group. The preliminary conclusions communicated to the Academy at the annual meeting of 1916 in my general report on the perturbations and tables of the minor planets discovered by James C. Watson are thereby fully verified.

Hygiea was discovered on April 12, 1849 by Gasparis. The most accurate of the eariler orbits computed was that by von Zech, which was based on the elements of d'Arrest and the general perturbations, developed by Hansen's method, by Jupiter, Saturn, and Mars, extending over eight oppositions. Since von Zech's death in 1864, his computations have been kept in the Recheninstitut at Berlin and were used by von Zeipel in connection with the application of his approximate perturbations for the Hecuba Group. Von Zech's computations have been continued from year to year at the Berlin Recheninstitut until 1873, up to which time they represented the motion of Hygiea with considerable accuracy. In the Berlin Jahrbuch for 1876 new elements by E. Becker are given. These were later corrected on the basis of oppositions extending from 1868 to 1874 with consideration of the special perturbations by Jupiter and Saturn. These elements have been carried forward with perturbations until 1898 and from that time on until 1920 without change of osculation. For 1920 the elements were improved by the Berlin Recheninstitut by a slight correction of the mean motion so that the last 14 oppositions appear to be represented with an accuracy of ±3′. In the mean time, however, observations extending from 1915 to 1918 had revealed increasing discrepancies between the observations and computed places. On October 9, 1918, this discrepancy amounted to nearly $2^1/_2°$ in right ascension and 1° in declination. These discrepancies have led to new orbits and ephemerides by Blondel, Maitre and Jehkowski, published in the Bulletin of the Observatory at Marseilles and to the latest revision of the elements by the Berlin Recheninstitut. These latter elements will no doubt represent observations at the next few oppositions with a satisfactory degree of accuracy, provided the laborious computation of the special perturbations and the correction of elements from opposition to opposition is kept up. Otherwise large discrepancies are bound to reveal themselves again in the course of a few oppositions. Even then, this piece-meal procedure does not produce a set of elements or a theory of perturbations which will represent the observations in all of the oppositions from 1849 to the present time and for at least an equally long interval in the future.

Aside from the enormous amount of computation of temporary value which is involved, it is necessary to secure frequent observations in order to secure the data for the successive improvement of the elements and perturbations. Thus, during the last ten years, more than 100 accurate photographic and visual observations of Hygiea have been published. These difficulties may now be wholly overcome by the application of von Zeipel's revised Jupiter perturbations of planets for the Hecuba Group and by the application of similar tables to other groups as originally proposed by Bohlin. As stated above, von Zeipel developed the perturbations of Hygiea from his original tables on the basis of von Zech's elements. With these perturbations he represented oppositions from 1849 to 1884. This representation showed a constantly increasing discrepancy which in 1884 had reached 23′ in right ascension and 9′ in declination. He therefore corrected the elements by the method of least squares and thereby succeeded in representing the observations satisfactorily until 1914. Discrepancies will hereafter rapidly increase in the opposite direction. The original discrepancies were not due so much to inaccuracy in von Zech's elements as to some inaccuracies in von Zeipel's tables for the Hecuba Group. This is clearly shown by the fact that von Zech's unchanged elements and our revised tables give as satisfactory a representation in the 37 years from 1884 to 1921 as in the preceding 35 years from 1849 to 1884, and that several experiments by Miss Glancy, the last published in 1918 in the Astronomical Journal Number 748, to remove outstanding differences between theory and observation by correcting von Zech's elements have led to invalid results, as in the case of von Zeipel's earlier attempt in the same direction.

Outstanding differences between theory and observation are of the same order of magnitude in the two intervals. In fact in 1917 the discrepancy was less than 1′ in both right ascension and declination, although in a very few of the long series of oppositions positive or negative differences as high as 10′ exist in right ascension. These differences however, are primarily due to perturbations by Saturn which so far have not been taken into consideration. These encouraging results on Hygiea clearly show that on the basis of the revised tables for the Hecuba Group and on the basis of reliable elements determined from a very few oppositions, perturbations may be developed for all planets of this group which will yield a satisfactory theoretical representation of observations for practical purposes. It is safe to say that the abandonment of the piece-meal method of computing special perturbations and correcting elements from opposition to opposition in favor of the group theory will release in the long run a considerable part of the efforts of the astronomers who are engaged in these laborious tasks.

175 Andromache was discovered by Watson on October 1, 1877. A preliminary report on the perturbations of this planet was rendered at the annual meeting of the Academy in 1916. At that time it was pointed out that as shown by the very accurate computations by Berberich at the Recheninstitut of the special perturbations of Jupiter and Saturn, the mean motion had constantly decreased from 617.7 in 1877 to 607.9 in 1910. During the last ten years the osculating mean motion has decreased but slightly. According to Miss Levy's development of the perturbations of Andromache by the revised tables this decrease in the osculating mean motion is due in the main to three long period terms of Jupiter's perturbations having a period of approximately 187, 94, and 62 years, with an amplitude of 23.3° for the perturbation of longest period. By introducing the effect of these long period terms into the mean motion, Dr. Thiele, who has made the necessary computations, and I have approximately reproduced the nearly progressive change in the osculating mean motion as computed by Berberich. By extending this process, we may predict that the minimum value of the osculating mean motion is nearly reached and that within the next few years it will begin to increase. A case of libration is therefore not involved as the mean motion will not pass through the value which is twice that of Jupiter. The maximum value will be reached between the years 2004 and 2016, after which it will again decrease. In fact, none of the cases of planets of the Hecuba Group so far examined

in this manner indicate the existence of a libration. According to all indications, the motion of these planets is therefore entirely stable. The very extensive and accurate computations of Berberich have furnished very reliable sets of osculating elements of Andromache. With Berberich's elements osculating for 1877 and Miss Levy's development of the perturbations on the basis of the revised tables, an observation for the year 1920 has been represented with a remarkable degree of accuracy, so that in this case, also, the process of computing the special perturbations from opposition to opposition and frequent observations may be abandoned for many decades to come.

It is hoped that the results for 10 Hygiea and 175 Andromache here presented will prove a determining factor in the methods hereafter to be applied by astronomers in deriving the approximate perturbations by Jupiter of planets of the Hecuba Group and of other groups with mean motion commensurable to that of Jupiter to which Bohlin's original group theory is applicable with proper modifications and extensions. These studies should form a most fruitful field of research in theoretical astronomy.

A METHOD OF DERIVING THE DISTANCE OF THE A-TYPE STARS

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Read before the Academy, April 25, 1922

Our knowledge of the distances of individual stars of the *A*-type of spectrum is obtained almost wholly from three sources.

1. Trigonometric parallaxes.
2. Dynamical, sometimes called hypothetical, parallaxes of binary stars.
3. Parallaxes derived from group motion.

The third method probably furnishes the most accurate values for all such stars as belong to well-recognized groups. In the case of the Taurus Group, for example, the distances are known with a high degree of precision. The trigonometric parallaxes are, of course, most valuable in the case of large values for which the probable error bears a relatively small ratio to the quantities measured. The dynamical parallaxes derived from binary stars with well-determined orbital motion are of excellent quality, but are affected to some extent by the uncertainty in the value of the mass-factor for the *A*-type stars which enters directly into the computation.

A list of 100 stars with distances and absolute magnitudes obtained by these methods has formed the basis of an attempt upon our part to investigate a possible correlation between absolute magnitude and the intensities of certain spectral lines similar to that which has been used successfully in the case of stars of the later types of spectrum. The material used includes the stars for which spectra have been obtained with the Cassegrain spectrograph at Mount Wilson, and the values of the parallaxes and absolute magnitudes have been taken from the lists of trigonometric parallaxes of various observers, an unpublished investigation by Russell on dynamical parallaxes, and the memoir of Rasmuson on moving clusters. Corrections have been applied to the dynamical parallaxes to reduce them to the values corresponding to the most probable value of the mass of the A-type stars.

	No.	M	M_{A7}
B_{9n}	9	0.83	+2.4
A_{0n}	14	1.09	2.4
A_{1n}	12	0.91	2.0
A_{1s}	4	0.15	2.4
A_{2n}	12	1.59	2.4
A_{2s}	7	0.51	2.2
A_{3n}	7	1.59	2.2
A_{3s}	8	1.18	2.3
A_4+A_5	11	2.10	2.6
A_6+A_7	11	2.24	2.3
A_8+A_9	10	2.58	2.3
$F_0+F_1+F_2$	4	2.82	2.3

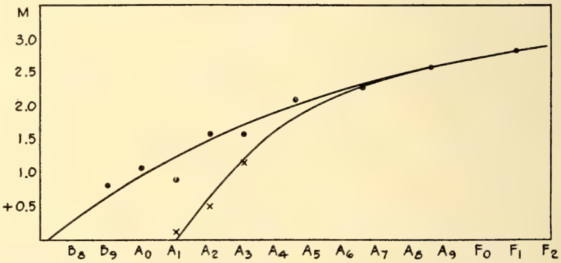

FIG. 1

Curves showing the relationship between absolute magnitude and spectral type for stars with parallaxes determined trigonometrically, dynamically and from group motions. Crosses refer to spectra with sharp lines A_1s, points to spectra with diffuse or nebulous lines as A_1n. In the present case this distinction is not carried into the more advanced sub-divisions,. MA_7 denotes the absolute magnitudes reduced to spectral type A_7 by means of these curves.

The starting point of the investigation and, as it proved, its most important feature, was an accurate determination of spectral type and the characteristics of the spectral lines. The stars were classified according to the Harvard scale as A_0, A_1, A_2, etc., and the spectra were designated *n* or *s* according as the lines are nebulous or sharp. This distinction is marked among the stars of the earlier subdivisions but becomes less certain at *A*5 or later. In addition to these two classes with sharp or nebulous lines is the small group of stars, of which α Cygni is the most prominent example, which are characterized by the Harvard observers as *c*-stars. These have exceptionally sharp and narrow lines and the enhanced lines as a rule are remarkably strong. In most of them the enhanced lines of strontium at λ4077 and λ4215 are very intense.

The following table gives the absolute magnitudes *M* of the stars arranged according to spectral type.

	NO.	M	MA_7		NO.	M	MA_7		NO.	M	MA_7
B9n	9	+0.83	+2.4	A2n	12	+1.59	+2.4	A4+A5	11	+2.10	2.6
A0n	14	1.09	2.4	A2s	7	0.51	2.2	A6+A7	11	2.24	2.3
A1n	12	0.91	2.0	A3n	7	1.59	2.2	A8+A9	10	2.58	2.3
A1s	4	0.15	2.4	A3s	8	1.18	2.3	$F_0+F_1+F_2$	4	2.82	2.3

An inspection of these results shows a steady increase in the values of the absolute magnitude with advancing spectral type and a marked difference in brightness between the stars of the same type according as the lines are diffuse or sharp. At type *A*4 and later no distinction is made as regard this latter characteristic except in the case of *c*-stars.

If these results are plotted and curves drawn through them we can from the curves read off the average absolute magnitude for each spectral type. These are as follows:

	DIFFUSE	SHARP		DIFFUSE		ARP
B8	+0.4		A5	+2.1		+1.8
B9	0.7		A6	2.2		2.1
A0	1.0		A7		2.3	
A1	1.2	0.0	A8		2.5	
A2	1.5	0.6	A9		2.6	
A3	1.7	1.2	F0		2.7	
A4	1.9	1.5	F1		2.8	
			F2		2.9	

Finally if the absolute magnitudes of the original groups of stars are all reduced to the same spectral type *A*7 by means of these curves we obtain the results given under MA_7 in the first table. These show a remarkable degree of accordance, the total range among the groups being from 2.0 to 2.6 with an average deviation from the mean of 0.1 magnitude.

Within the groups themselves the probable error for an individual star is given below. This assumes that there is no dispersion in absolute magnitude around the mean value for a group of stars of a given spectral type with definite characteristics to their lines. That the observational results indicate this dispersion to be small is shown by the moderate size of the probable errors in spite of the fact that these are in some cases probably increased greatly by errors in the trigonometric parallaxes.

	NO.			NO.	
A9n	9	±0.48	A3n	7	±0.54
A0n	14	0.46	A3s	8	0.07
A1n	12	0.75	A4+A5	11	0.49
A1s	4	0.16	A6+A7	11	0.69
A2n	12	1.02	A8+A9	10	1.24
A2s	7	0.67	F0+F1+F2	4	0.91

The result of the comparison is to indicate that for normal *A*-type stars the absolute magnitude and parallax can be determined with a precision quite comparable with other methods from an accurate observation of spectral type and the characteristics of the spectral lines. For the *c*-stars, at least in such cases as α Cygni, this is not the case, but these stars appear to show in their spectra the phenomena connected with abnormal intrinsic brightness, and like the Cepheid variables of the *F*-type which they resemble noticeably there is every reason to believe they require a separate method of reduction.

Of the application of these results to individual stars it is possible to refer only very briefly but the following results may be of interest.

	TYPE	COMP. π	OBS. π
α Lyrae	A1s	0.09	0.10
ζ^1 Ursae Majoris	A2s	0.044	0.045
Taurus Group (26 stars)	A0-F	0.024	0.023
Praesepe (4 stars)	A2-A6	0.011	0.011

The close relationship of these results to certain observations by other investigators is of interest. It was found by Hertzsprung and later confirmed by Kohlschütter[2] that the stars in the Taurus cluster show approximately a linear relationship between color index or spectral type and absolute magnitude. These results deal in general with the larger spectral classes and later types and not with the subdivisions of a single type as is the case here.

In a recent investigation of the luminosity of the *A*-type stars, Lindblad[3] found that the intensity of the continuous spectrum in a region between $\lambda3895$ and $\lambda3907$ was less for the intrinsically fainter stars of this type than for the brighter stars. This effect he ascribes to the increased intensity of the wings of $H\zeta$ and the greater width and intensity of some of the other lines, especially those of iron and silicon, in this region. In other words the spectra of the fainter stars are diffuse as compared with those of the brighter stars, a result in agreement to that found in this investigation.

There seems to be little doubt that great intrinsic luminosity is associated with sharp and narrow lines in all spectral types and that this is due to the low density in the atmospheres of such stars. In stars of various spectral types, such as β Orionis, α Cygni, the Cepheid variables and α Orionis this effect is very marked, and the change in the character of the lines of the variable star *o* Ceti from maximum when they are well-defined to minimum when they are diffuse and vague is an excellent illustration of the same phenomenon.

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[1] Hertzsprung, *Astron. Nachr.*, **209** (115).

[2] Kohlschütter, *Ibid.*, **211** (289).

[3] Lindblad, *Astroph. J.*, **55** (85).

THE LARGER RESULTS OF 20 YEARS OF SOLAR RADIATION OBSERVATIONS

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Read before the Academy, April 24, 1922

In Volume 4 of the *Annals of the Astrophysical Observatory* are given the results obtained from 1912 to 1920, but the discussions go back to the beginning of the investigations of solar radiation at the Smithsonian Institution in 1902. Where so many details are taken up it is a little hard to keep in mind the broad outlines, and we propose to give a general view of the work in what follows.

We measure the solar heat received by the earth; determine the limits of its variability; study the distribution of solar radiation in the spectrum, and its intensity over the solar disk; and we investigate the properties of the terrestrial atmosphere in relation to the passage of solar rays to the earth's surface and the escape of earth rays to space.

When the work was begun in 1902, great diversity of results prevailed. The black bulb in vacuo thermometer, Pouillet's pyrheliometer, Tyndall's, Violle's and Crova's actinometers, and the older form of Ångström's compensation pyrheliometer, differed through a range of at least 50 per cent in their indications. Values of the "solar constant" or intensity of solar radiation outside our atmosphere at mean solar distance ranged from 1.76 to 4.0 calories as given in the best text-books. Nothing was known as to the limits of the solar variability.

1. *Standardization and perfecting of methods and apparatus.*—While the methods we have used are Langley's they have been reduced to a finished system of every day procedure. The theory, practice, sources of error and details needful to accurate results in solar constant observing have been worked out and published in Volumes 2, 3, and 4 of the *Annals of the Astrophysical Observatory.*

Two forms of standard pyrheliometer have been devised, perfected and used to establish the scale of solar radiation measurement. Both types of instrument have hollow chamber or "black body" receivers of radiation, and in both test quantities of electrically introduced heat have been measured with close agreement.

A secondary instrument, called the silver-disk pyrheliometer, has also been perfected. About 30 standardized copies have been purchased by observers in many countries of the world. Thus the standard scale of radiation has been widely diffused. Our measurements are approximately 2 per cent higher than those of the system of Ångström with its latest corrections.

Among other instruments developed or improved for the work are the 2-mirror coelostat; the reflecting galvanometer; the vacuum bolometer for which a full mathematical theory was derived; the pyranometer, an instrument for measuring the radiation received from the daylight sky; the honeycomb pyranometer or melikeron, a "black-body" instrument for observing nocturnal radiation; the ballon pyrheliometer, 5 copies of which attached to sounding balloons, were raised to great altitudes in 1913 and 1914 for observing the solar radiation very high up in the atmosphere; the slide wire extrapolator and special plate measuring instrument, both used for the reduction of observations of the solar constant of radiation.

2. *Value of the Solar Constant.*—About 2000 observations of the intensity of solar radiation outside the atmosphere at mean solar distance, otherwise called the "solar constant," have been made according to the high and low sun method of Langley as systematized and perfected by use. These measurements have been made at Washington (sea level); Bassour, Algeria (1,160 meters); Hump Mountain, North Carolina (1,500 meters); Mt. Harqua Hala, Arizona (1,729 meters); Mt. Wilson, California (1,735 meters); Calama, Chile (2,250 meters); Montezuma, Chile (2,900 meters?); and Mt. Whitney, California (4,420 meters). The circumstances of observation have been widely varied, ranging from days of high humidity in summer at Washington to less than a millimeter of precipitable water as in the atmosphere above Mt. Whitney; at temperatures from those of the Sahara Desert in Algeria to $-25°$ C. at Hump Mountain, North Carolina; and clearness of the atmosphere, from the deep blue which prevails right up to the sun's disk on Mt. Whitney and Mt. Montezuma down to the white dusty sky which prevailed in the year 1912 at Mt. Wilson and in Algeria after the eruption of the volcano Mt. Katmai in Alaska.

These varied circumstances appear to produce very little influence on the solar constant results, and the mean value, 1.94 calories per square centimeter per minute, is closely confirmed by the observations of the balloon pyrheliometer raised to an elevation of nearly 25,000 meters where the atmospheric pressure is reduced to one twenty-fifth of its value at sea level.

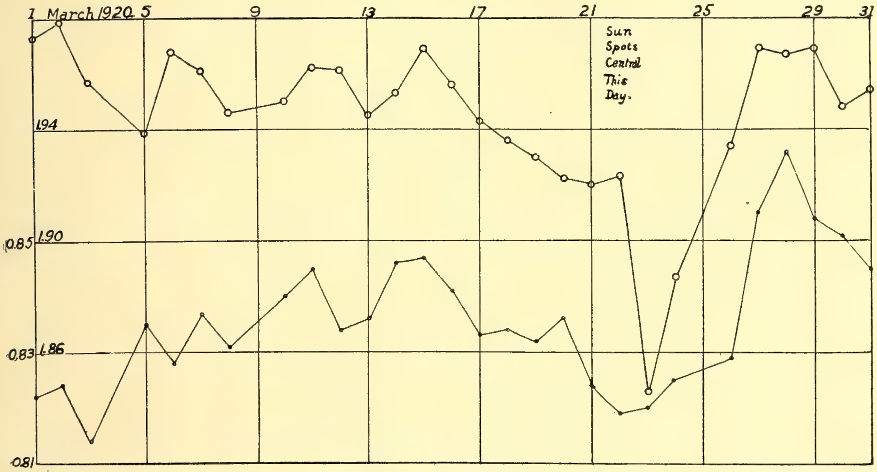
We believe this value, 1.94 calories per square centimeter per minute, may be regarded as the datum mark for this epoch with which may be compared in future centuries the solar radiation at that time prevailing in order to determine what, if any, secular changes of it have occurred.

3. *Solar Variation.*—To state the negative first, our observations have shown that the sun is not, like some of the other stars, widely variable but has at any rate not been found to vary in excess of 12 per cent in the last 20 years. In general its variations do not exceed 5 per cent in a month.

On the other hand, solar variations of from 1 to 5 per cent appear to occur at irregular intervals and with irregular magnitudes of variation;

the observed range often reaches 2 per cent within a period of ten days, and not infrequently, as, for instance at the time of the great central sun-spot group of March 22, 1920, a change of 5 per cent within 10 days is noted. The extreme range of values which has been observed by us since 1902 is approximately 12 per cent.

Long period solar variation attending the changes of solar activity, as evidenced by sun-spots, prominences, and other visible solar phenomena, appears to reach about 3 per cent for 100 Wolf sun-spot numbers. Higher solar radiation prevails at times of higher solar activity.



Solar constant and atmospheric transmission affected by great central sun-spot group, March, 1920.

Lower curve, atmospheric transparency wave-length 0.511 microns, Calama, Chile.

If *depression* of atmospheric transmission were erroneous, the result would be *increase* of solar-constant values. Apparently bombardments of earth by solar ions attending central passage of sun-spot group diminish transparency of the earth's atmosphere. This effect is frequently noted.

4. *Confirmatory Evidences of Solar Variation.*—(*a*) By studies of the distribution of radiation in the telescopic image of the sun's visible disk, fluctuations of the contrast of brightness between the center and edge of the sun have been observed for many wave-lengths. These fluctuations have complex correlation with the observed variations of the solar constant of radiation.

(*b*) In the distribution of radiation in the solar spectrum outside the atmosphere, high values appear at the shorter wave-lengths, as compared with the longer ones, when the solar radiation values are high.

(*c*) Solar radiation values themselves have been carried on at stations many thousands of miles apart and have checked one another in their indications of solar variability.

(*d*) From a rather small number of observations by Guthnick on the brightness of the planet Saturn, it appears that equal fluctuations of the brightness of the planet attend observed fluctuations of the radiation of the sun. This result, however, requires that allowance shall be made for the rotation of the sun to take into account the difference in heliographic longitude between the earth and Saturn.

(*e*) Various observers, notably Clayton, Arctowsky, Helland-Hansen, and Nansen, have published results showing the correlation between terrestrial temperatures, and other meteorological phenomena, and our observed variations of the sun.

(*f*) Bauer found a correlation between solar variation and the variations of terrestrial magnetism.

5. *Atmospheric Transparency.*—We have determined the transparency of the earth's atmosphere to solar radiation at stations ranging from sea-level to 4,420 meters for nearly 40 wave-lengths. We have observed the dependence of the atmospheric transparency on humidity and volcanic dust, and its variation from month to month, and from year to year. Since atmospheric transparency is associated, by Rayleigh's theory of the blue of the sky, with the number of molecules per cubic centimeter of air, our results have led to an independent value of this constant for which Mr. Fowle finds 2.70×10^{19} molecules per centimeter cube in air at standard temperature and pressure.

6. *Brightness of the Sky.*—Measurements have been made of the brightness of the sky by day, as a whole and in parts, and the variation of it from sea-level to 4,420 meters, as well as the effects upon it of the presence of clouds.

Mr. Aldrich took advantage of the presence of the Balloon School at Arcadia, near Mt. Wilson, to observe the reflecting power of clouds as formed there in a great level sheet some 500 meters high. From his results, it appears that if the earth were completely enshrouded in a smooth layer of cloud its reflection would be approximately 77 per cent.

7. *Long-wave Ray Investigations.*—Mr. Fowle carried on spectrobolometric investigations of the rays of the Nernst glower as affected by the humidity and carbon dioxide prevailing, in very long absorbing tubes at atmospheric pressure and temperature. These measurements covered the spectrum in the region of the earth's emission up to a wave-length of nearly 20 microns. The investigation included the distribution of the rays in the solar spectrum in this region. Many measurements have been made also on the absorption and reflection of liquid water, and many other terrestrial substances and of their degree of transparency to long-wave rays.

9. *Distribution of Radiation over the Sun's Disk.*—It was previously known that the brightness of the sun's disk falls off towards the edges, but

these measurements have been made in detail for many different wave-lengths and with the very high degree of precision necessary to observe the fluctuations of contrast of brightness from time to time. The measurements extended beyond the visible spectrum both towards the ultra-violet and towards the infra-red, and showed a striking change of distribution along the sun's diameter, depending on the difference of wave-length.

10. *Solar Theory.*—These solar radiation measurements, including those of the solar constant which have led to a value of the sun's effective temperature, those on the distribution of light over the sun's disk, and those on the transmission of the terrestrial atmosphere considered in conjunction with the known results of others on the details of the solar spectrum, have led to a theory of the sharp boundary of the sun and other things relating thereto which appears more and more to gain acceptance.

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*REPLANTATION OF ENTIRE LIMBS WITHOUT SUTURE OF VESSELS**

BY WILLIAM S. HALSTED

MEDICAL SCHOOL, JOHNS HOPKINS UNIVERSITY

Read before the Academy, April 24, 1922

The experiments were undertaken with view to determining the amount and duration of the swelling which would result from the division of all the vessels and lymphatics of a limb, our interest in the subject arising from the desire to obtain experimental support of my contention that permanent swelling of the arm following operations upon the axilla is probably invariably due to infection, an infection often so slight as to escape the observation of the surgeon. It is almost a daily experience of active surgeons to see swollen arms as result of radical operations for cancer of the breast. The swelling is frequently so great as to cause suffering and disability from the weight of the arm, and occasionally patients have consented to amputation of the entire limb for the condition which surgeons have been powerless to relieve. The ordinary swelling is universally attributed to the removal of the axillary lymphatics and veins, and some surgeons have advocated operations less thorough, believing that their results as regards the swelling of the arm have been better when the axilla was less meticulously cleaned.

Almost from the beginning of our radical operations for cancer of the breast I have been impressed with the fact that infection, however slight, was likely to be followed by more pronounced swelling of the arm and that the largest arms were noted as a rule in cases in which the inflammatory reaction and the axillary induration were greatest. We assumed, naturally, that the excessive swelling must be due to the blocking of lymphatics and possibly veins uninjured by the knife.

About thirteen years ago a modification of the operative procedure was made, particularly in the manner of closing the wound, and thereafter, to our surprise, swollen arms became almost a thing of the past. I say "to our surprise," for the modifications in the operation were made primarily merely in the hope of securing greater freedom of motion at the shoulder joint.

Now inasmuch as the destruction of lymphatics and veins was identically the same in the modified as in the original procedure, we concluded that the most thorough excision of the axillary contents could not alone produce swelling of the arm, and that whatever the predisposing cause might be, the ultimate factor in the production of the swelling must be something which had been eliminated in the newer procedure.

It is a common observation that a moderately swollen arm may after an interval of a year or more become markedly increased in size. This augmentation is generally and quite properly attributed to a recurrence of the disease. Occasionally, but rarely, I have observed an arm that for a year or more after the operation had maintained approximately its normal size swell quite acutely. The swelling might follow closely upon constitutional symptoms—nausea, malaise, chill and fever—and be coincident with a slight or even bright blush of the skin of the shoulder, arm, and perhaps chest and forearm.

For example, in May, 1920, a patient upon whom I had performed a radical operation for mammary cancer in November, 1916, consulted me in regard to an acute swelling of the arm on the operated side. The arm had maintained its normal size prior to an "attack" in February, 1920, in the course of her convalescence from influenza. This attack as she terms it was ushered in by nausea, a chill and high fever. The arm promptly began to swell and there appeared "redness in streaks" from the shoulder to the wrist; in a few days the redness was diffuse and the swelling of the arm had become distressingly great; the hand, she said, resembled a "boxing glove," and pressure on it with the fingers produced "deep pits." In eight or ten days the redness had vanished and the swelling was decreasing. During the three months prior to this her second admission to the Johns Hopkins Hospital, there had been a less rapid reduction in the size of the affected (left) limb, which about the middle of the arm measured in circumference 9.5 cm. more than the other, and at the middle of the forearm, 3.5 cm. more. No glands were palpable above the clavicle, and there seemed to be no abnormal fullness or resistance below it. In the skin at the outer-upper edge of the grafted area there was recurrence of the carcinoma—two nodules, not ulcerated, one about the size of a filbert, the other smaller than a split pea.

The immediate cause of the swelling was undoubtedly the local infection, for during the three years and three months prior to this and subsequent to the operation there had been nothing to indicate a blocking by recurrent disease of the lymph channels.

Eight days after the second admission (June 1, 1920), the patient was operated upon by Dr. Mont Reid, our resident surgeon. The second and a part of the third portion of the axillary vein were found to be completely occluded by the new growth, which was continuous with the larger of the two cancerous nodules in the skin. The axillary vein and the recurrent growth were excised in one piece, the disease being given a wide berth. On examination of the specimen it seemed quite clear that the vein had been invaded from without, and had surely been occluded long before the attack of infection. After this operation the swelling rapidly subsided and in the course of two months the arm had almost regained its normal dimensions.

In this case, as in a number of others observed in our clinic, the occlusion of the axillary vein plus the excision of the axillary lymphatics was not followed by any swelling of the arm.

I have under observation several patients who each year following the operation have had one or two or more attacks of the kind described, each attack being followed by an increase in the size of the arm.

Only four days ago I received an announcement of the death of a patient whose story subsequent to an operation for cancer of the breast sustains the view that swelling of the arm may be due to infection even when local and general signs of inflammation are wanting. This patient was operated upon by me five years ago. About nine months after the operation she consulted me in regard to a rather acute but inconsiderable swelling of the arm on the affected side, which theretofore had not been swollen. She could not recall having had constitutional symptoms and was quite sure there had been no inflammatory redness of the arm coincident with first appearance of its enlargement. From the time of this interview until her death, four years and three months later, there had been no augmentation of the swelling. She died of carcinomatous metastases in the bones. There was no recurrence of the disease either locally or in lymphatic glands which could have affected the circulation of the arm. Hence the swelling which occurred nine months after the operation could not have been due to the blocking of lymphatics by cancer.

Operations upon the glands of the groin may be followed by elephantiasis of the scrotum. One patient in this category was operated upon by me about forty years ago. During all this period he has each year had mild local attacks of an infection resembling erysipelas and with each attack a slight, transient, increase in the elephantiasis.

Now if redness or even a faint blushing of the arm always accompanied the advent of the swelling which occurs after an interval the discovery would undoubtedly have been made long ago of the relationship of infection to the swelling of the arm. But an infection quite negative in external manifestations other than swelling may I believe exist and suffice to produce the further blocking which in my opinion is essential to the production of the swelling.

In a word, then, I have thought it unlikely that the excision of the axillary contents, lymphatics and veins, could alone be responsible for the swelling of the arm which is observed after the radical operation for cancer of the breast in non-recurrent cases.**

Thirty-five years ago (in 1887) I successfully transplanted the hind leg of a dog from one side to the other, leaving however the main artery intact for a few days—until union between the muscles and other divided tissues had taken place. And in more recent times (1907) Carrel, with the aid of his vascular suture, made the remarkable discovery that the leg from one dog can be transplanted to another.

The operative work in the current experiments has been done by my assistants, Dr. F. L. Reichert, Dr. Mont Reid and Dr. C. Y. Bidgood. The arterial and lymphatic injections, the preparation of the specimens, the X-ray studies are all the work of Dr. Reichert.

At the outset of our experiments I had it in mind merely to determine the amount and duration of the swelling which would follow the division of all the lymphatics and veins of the thigh. At the primary operations all the soft tissues except the femoral artery and vein and the sciatic nerve were divided and immediately reunited by suture. On the second day after the replantation the femoral vein was tied in two dogs; in one of these gangrene promptly followed the ligation. Then on the fourth day after replantation the experiment of ligating the femoral vein was twice made; gangrene resulted in one of these dogs. Ligations on the 5th, 6th, 7th, 9th, and 11th days and at later periods have not been followed by gangrene; and if performed as late as the 9th day, by little or no increase in the size of the limb. Replantation and simultaneous ligation of the femoral vein was invariably followed by gangrene—6 cases.

In one case the femoral artery and vein were ligated simultaneously with the replanting—gangrene followed. Both femoral vessels may be safely ligated on the 6th day, and probably earlier.

In one instance the replantation was successful notwithstanding the fact that the femoral artery and vein had become totally obliterated by ligation of these vessels 7 months previously.

As the possibility of an anastomotic circulation by way of the bone and sciatic nerve naturally occurred to us, Dr. Reid, in one dog, divided these structures as well as the others. The replantation was successful.

To Dr. Reichert and Dr. Bidgood belongs the credit for the discovery that replantation of an entire limb may be possible when and even when the main artery and vein have been previously ligated. The story of the successful case is as follows: November 1, 1920, Dr. Reid replanted the left hind leg of a dog. Ten days later Dr. Reichert ligated the femoral artery and vein of the opposite (right) leg for the purpose of making X-ray studies of the anastomotic circulation. On June 6, 1921, 7 months thereafter, Dr. Bidgood and Dr. Reichert amputated through the right thigh of this dog—bone undivided, having forgotten that the main vessels of this leg had been previously tied. They were surprised therefore to find that these vessels had become converted into fibrous cords and naturally feared that the replanted leg would become gangrenous. Finding later that the leg would live, we regretted that the obliterated vessels had not been severed. Hence Dr. Reichert excised a portion of each of the fibrous cords and examined them microscopically. Not only had the lumina become obliterated but the strands consisted of nothing but white fibrous tissue, the muscle and elastic tissue of the coats having disappeared. Whatever may be the explanation of this successful replantation in one act, the result is remarkable—almost beyond belief. This experiment will of course be repeated, many times if necessary, and Dr. Reichert has made the preliminary ligation of the femoral artery and vein in a number of dogs with the intention of practising the amputation and replantation at various periods thereafter.

Surgeons can best comprehend the perfection of technique required for conducting such experiments. Were it not for the fact that well nourished tissues can take care of many organisms few wounds would heal by first intention. But in the replantation experiments unusual precautions have to be taken for the reason that a devitalized limb presumably has little power to resist infection. Of the two surfaces which we are asking to unite, one is for a time quite dead if all the structures have been divided. When the artery and vein have been left for secondary ligation little imperfections in technique need not be disastrous. The impossibility of having up to the present time an absolutely perfect technique may possibly account in part for the fact that the replantations have not succeeded when all the structures have been divided except in the one instance. It is, however, astonishing, I may say hardly believable, that the complete replantation should have been successful even after preliminary ligation of the artery and vein. Having once been accomplished, an occasional repetition of the success may safely be predicted. With a technique still further perfected we might find that without preliminary ligation of the main vessels a complete replantation is occasionally possible.

After 16 days and probably earlier the new lymphatics crossing the scar are demonstrable by injection. This is perhaps the most satisfactory functional proof we have of the regeneration of lymphatics.

On the 6th day the femoral artery and vein may, as stated above, be safely ligated. Hence on the 6th day or earlier the entire circulation of the leg is carried on through the newly formed vessels—lymphatic, venous, arterial.

On the 12th day the swelling, which increases during the first week after replantation, has about disappeared.***

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* The complete paper with illustrations will be published elsewhere by Dr. F. L. Reichert.

** I was greatly pleased a few weeks ago to have my attention called by Sir Almroth Wright to the following paragraph in his paper entitled: "Introduction to Vaccine Therapy:" "Lastly—for it is impossible to mention all the points, streptococcic infection would appear to be responsible for that swelling of the arm so often seen after amputation of the breast—that swelling which finds, as we are told, a perfect explanation in the 'mechanical blocking of the lymphatics.' " (Nelson Loose Leaf Medicine, 1920.)

*** Lantern pictures illustrating the author's latest method of operating for cancer of the breast, of the dogs at various periods after the replantations and of skiagrams of the injected newly formed arteries were exhibited.

DEGENERATION AND REGENERATION OF THE LATERAL-LINE ORGANS IN AMIURUS NEBULOSUS

BY MARY H. CHAMBERS

ZOÖLOGICAL LABORATORY, RADCLIFFE COLLEGE

Communicated May 31, 1922

If the nerve innervating the taste buds in the barbels of the catfish is cut, the taste buds degenerate (Olmsted, 1920b) in ten to fourteen days. Regeneration of the taste buds follows the regeneration of the nerve and is completed in about sixty days. The lateral-line organs are related anatomically to the lateral-line nerve much as the taste buds are to their nerve. Do lateral-line organs degenerate and regenerate as taste buds do?

The lateral branch of the vagus nerve innervating the lateral-line organs was severed in Amiurus. The sense organs are conspicuous structures and normally the sensory and supporting cells are easily distinguished. The first indications of degeneration appeared on the fourth day after the operation at which time the organs began to lose their distinct appearance. The sensory cells decreased in size and disappeared, first in the central part of the organ, where the nerve enters it, and later peripherally. Fragments of sensory cells may be detected for a few days longer, as degeneration in the single sense organ progresses. The supporting cells are more persistent, but by the end of two weeks the whole organ has usually disappeared.

Regeneration is more complicated than degeneration. If regeneration of the sense organs accompanies regeneration of the nerve, one would expect to find the first regenerated sense organ in the region of the regenerated nerve. At fifty-six days there was evidence of regeneration in the sense organs and it occured very near the region of the regenerating nerve. Following the lateral-line posteriorly beyond the cut the first two sense organs in this region had not regenerated, but on the site of the third sense organ regeneration of the supporting cells had occured. Next followed a degenerated sense organ after which came a regenerated one, indistinct in outline, but with three or four sensory cells. Continuing beyond this there were no more regenerated organs.

Ninety-eight days after cutting the nerve regenerated sense organs occured in consecutive positions. The sense organs regenerate element by element as they degenerated. The supporting cells differentiate before the sensory cells do. The completely regenerated sense organs appear quite normal.

From the above data it may be concluded that degeneration of the lateral-line organs in Amiurus occurs within fourteen days after cutting the lateral nerve and that regeneration of these organs takes place after fifty-six days.

Olmsted, J. M. D. 1920a. The nerve as a formative influence in the development of taste buds. *J. Comp. Neur.*, **31**, (465–468).

Olmsted, J. M. D. 1920b. The Results of Cutting the Seventh Cranial Nerve in Amiurus nebulosus (Lesueur). *J. Exp. Zoöl.*, **31**, (369–401).

THE ISOLATION OF PYRIMIDINES FROM THE NUCLEIC ACID OF TUBERCLE BACILLUS

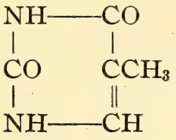
By Treat B. Johnson and Elmer B. Brown

Department of Chemistry, Yale University

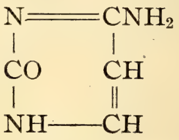
Communicated May 23, 1922

This paper is a preliminary announcement of results which have been obtained in an investigation of the nucleic acid of tubercle bacillus. This research was supported in part by funds provided for the work by the National Tuberculosis Association. The raw material employed in the research was obtained through the coöperation of Dr. Esmond R. Long of Sprague Institute, Chicago, Illinois.

With the quantity of tubercle bacillus placed at our disposal it has been possible to isolate the nucleic acid in sufficient quantity for hydrolysis and to make an accurate determination of the pyrimidines contained therein. We find that this nucleic acid is characterized by the fact that it does not contain uracil. At least we found no evidence of its presence in the product which we investigated. We have succeeded, however, in isolating in a pure condition the two pyrimidines, thymine I and cytosine II.



I



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Both pyrimidines were obtained in sufficient quantity for analysis and their constitution further established by the color reactions[1] characteristic of these two compounds, respectively. The two products isolated were compared with the pyrimidines obtained by synthesis and found to agree in every respect in chemical and physical properties. The further study of the chemistry of tubercle bacillus is in progress and the results of the investigation just finished will soon be published in The American Review of Tuberculosis and the Journal of Biological Chemistry.

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[1] Wheeler and Johnson, *J. Biol. Chem.*, **3**, 1907 (183). Johnson and Baudisch, *J. Amer. Chem. Soc.*, **43**, 1921 (2670) and *Ber.*, **55**, 1921 (18).

THE RELATIVE SENSITIVITY OF THE EAR AT DIFFERENT LEVELS OF LOUDNESS

By DONALD MACKENZIE

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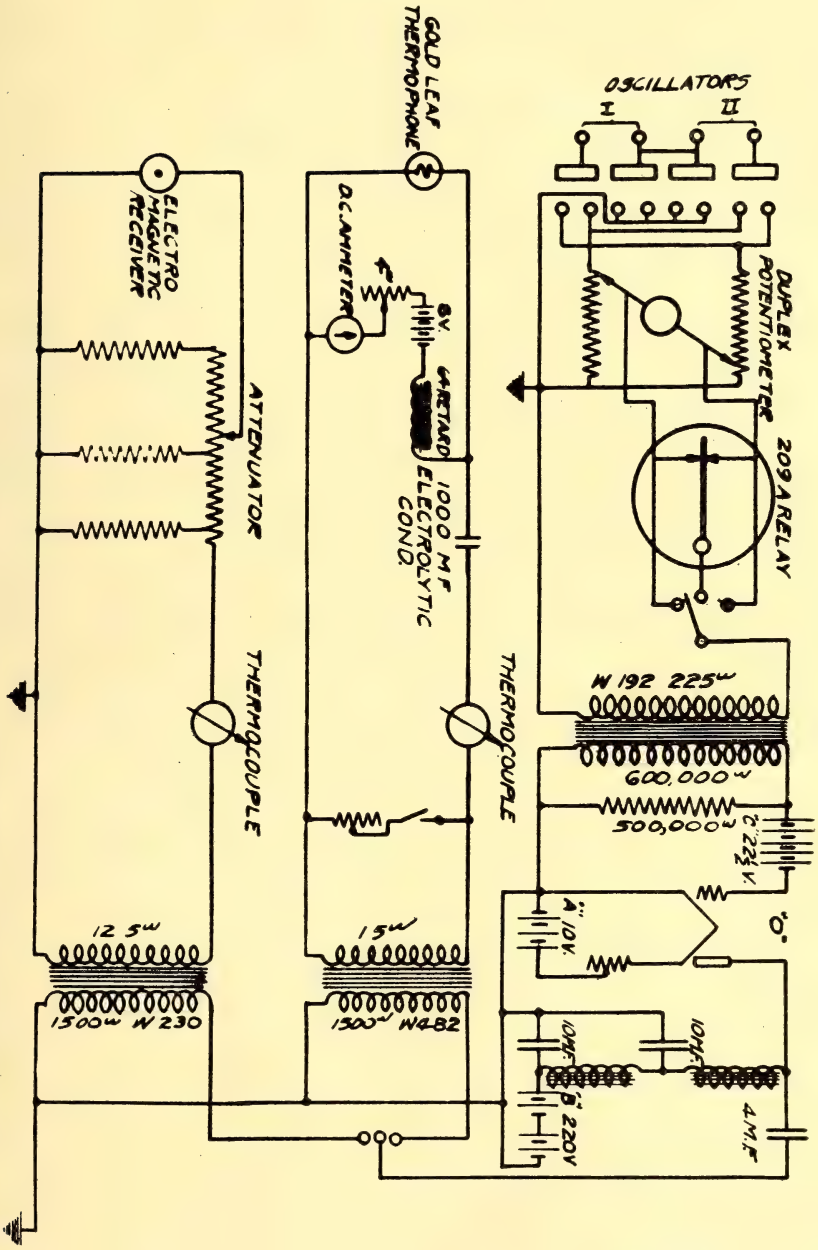
Read before the Academy, April 25, 1922

The threshold sensitivity of the ear for tones of various frequencies has been studied by many investigators. In order to determine the relative sensitivity at higher intensity levels, an alternation phonometer has been devised which permits balancing two tones of different frequency for equality of loudness.

The apparatus (fig. 1) consists of two vacuum tube oscillators, independently controlled, which through a relay commutator alternately supply current to a sound generator (either thermal or electromagnetic). The commutator is a polarized telegraph relay, driven at a suitable speed from another such relay. The commutating relay is adjusted to give a contact time of 1/25 second on each side, with a silent interval of 1/1000 second required for armature travel between contacts. This device differs in principle from the flicker photometer in that fluctuations in intensity are not caused to disappear and subjective fusion of the tones compared does not take place.* The interval (2/1000 second) through which auditory sensation persists after cessation of the physical stimulus



CIRCUIT OF THE ALTERNATION PHONOMETER

FIG 1

is so short that we cannot use the flicker principle as applied in photometry. Accordingly, the loudness balance is found when the flickers of the two tones are judged to be equally conspicuous.

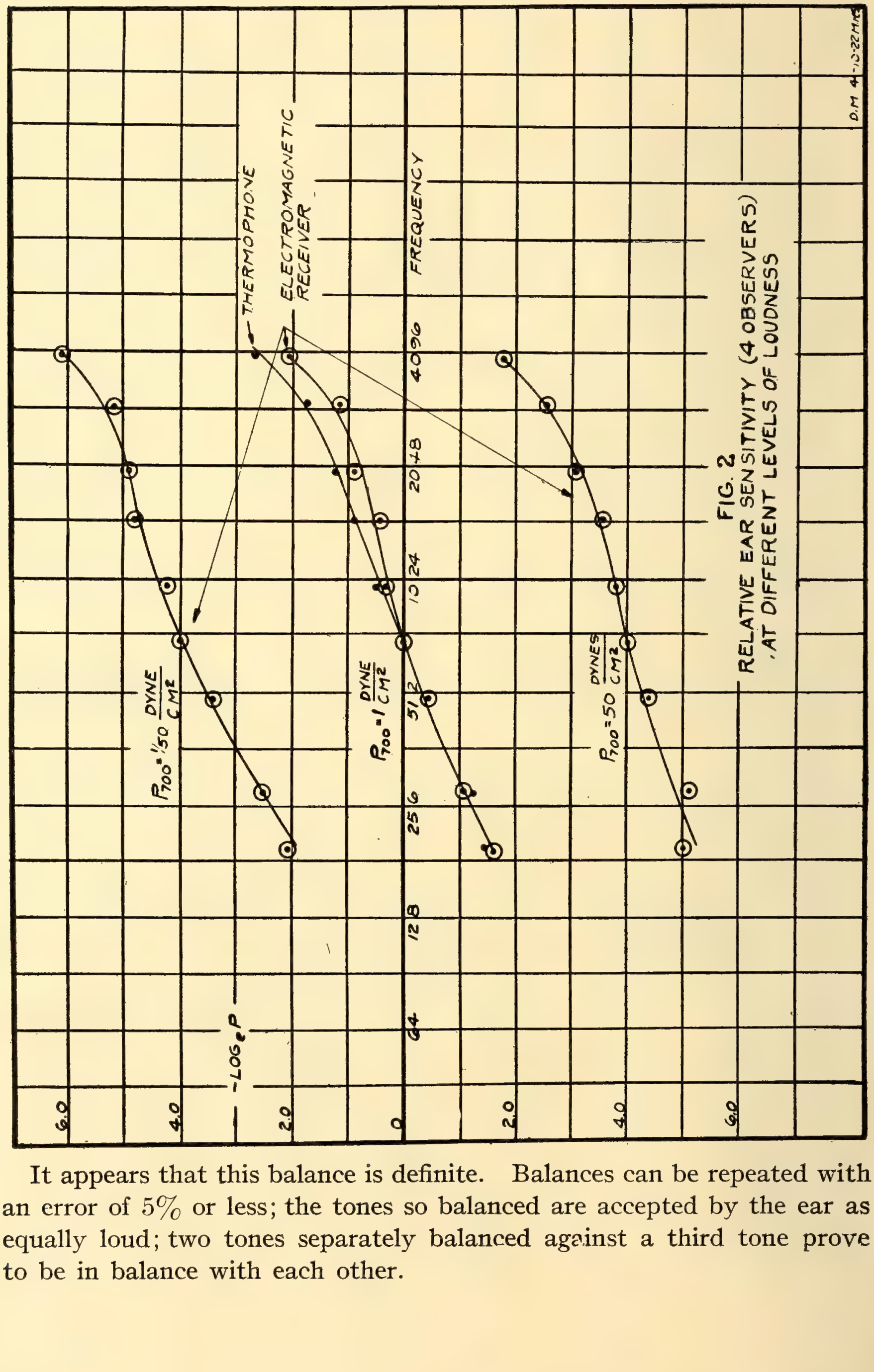


FIG. 2
RELATIVE EAR SENSITIVITY (4 OBSERVERS)
AT DIFFERENT LEVELS OF LOUDNESS

It appears that this balance is definite. Balances can be repeated with an error of 5% or less; the tones so balanced are accepted by the ear as equally loud; two tones separately balanced against a third tone prove to be in balance with each other.

Twenty observers (ten men and ten women) using each one ear made observations with the thermal receiver over a range of frequencies from 100 to 4000 cycles per second, and in each comparison over a range of 10 to 1 in acoustic pressure. Their results are expressible in a simple logarithmic law: $\log P_1 = A + B \log P_2$, where P_1 and P_2 are the alternating pressures of two tones phonometrically balanced. This result is concordant with the simple formulation of Fechner's law.

The simple statement of Fechner's law holds only over the range of intensities throughout which the least perceptible stimulus increment is constant. Since the thermal receiver did not cover a wide range (the ear readily accomodates to acoustic pressures between 1/1000 and 1000 dynes per sq. cm.) the investigation was extended over wider limits of loudness, substituting an electromagnetic receiver as sound generator.

Only four observers took part in this work. Some consideration was devoted to the types of curves to be expected if the ear is presumed to resemble the eye in its variation in response to varying values of the stimulus. The expectation was that two types of curves showing the relation between the alternating pressures of the two tones compared would be met with; sensibly straight throughout, and curved in a specific way.

Tones of the following frequencies were compared directly with the 700 cycle tone: 200, 300, 500, 1000, 1500, 2000, 3000 and 4000 cycles. At levels of loudness from 1 to 10 napiers above the threshold (one napier corresponds to a pressure ratio of e to 1) phonometric comparisons were satisfactorily made. The results were as anticipated: of 32 curves, 25 were sensibly straight with slope unity over the whole range, 7 were curved oppositely in upper and lower halves.

The acoustic output of the receiver when held against the ear cannot be directly measured. Systematic errors, therefore, affect the receiver calibration obtained under artificial conditions. Nevertheless, these errors are the same at all intensities, and the predominance of straight phonometric lines of unit slope is evidence for the conclusion that the relative sensitivity of the ear is practically invariable over the range from near the threshold to near the level of painful loudness.

The individual differences among observers are smaller at intermediate levels than at the threshold, and no significant difference is found between the ears of men and those of women. Fig. 2 exhibits the relative ear sensitivity derived from the observations of 4 observers using the thermal and the electromagnetic receivers.

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* A flicker phonometer similar in principle to the flicker photometer has been patented by C. M. Heck: U. S. Pat. No. 1,356,359; Oct. 19, 1920 "Method and Apparatus for for Comparing Sounds."

NORMAL COÖRDINATES FOR THE GEOMETRY OF PATHS

By. O. Veblen

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Read before the Academy April 25, 1922

1. The normal coördinates introduced by Riemann have been of the greatest utility in a variety of researches in Riemann geometry and, are likely to be important in the theory of relativity. An analogous coördinate system is fundamental in what Professor Eisenhart and I have called the Geometry of Paths (Vol. 8, p. 19 of these PROCEEDINGS) i.e., in the theory of the differential equations.

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{1.1}$$

in which

$$\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha} \tag{1.2}$$

the Γ's being functions of the variables, x^1, x^2,. . .x^n, and the paths being the curves which satisfy (1.1). The purpose of the present note is to define the new normal coördinates, to study a set of tensors connected with them, and to obtain a set of identities. Some of the formulae are believed to be new even for those manifolds in which the geometry of paths reduces to the Riemann geometry.

2. From the differential equations (1.1) we obtain by differentiation a sequence of differential equations.

$$\frac{d^3x^i}{ds^3} + \Gamma^i_{\alpha\beta\gamma}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \tag{2.1}$$

$$\frac{d^4x^i}{ds^3} + \Gamma^i_{\alpha\beta\gamma\delta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\frac{dx^\delta}{ds} = 0 \tag{2.2}$$

and so on, in which

$$\Gamma^i_{\alpha\beta\gamma} = \frac{\partial\Gamma^i_{\alpha\beta}}{\partial x^\gamma} - \Gamma^i_{j\beta}\,\Gamma^j_{\alpha\gamma} - \Gamma^i_{\alpha j}\,\Gamma^j_{\beta\gamma} \tag{2.3}$$

and in general

$$\Gamma^i_{\alpha\beta\gamma\ldots\xi\eta} = \frac{\partial\Gamma^i_{\alpha\beta\gamma\ldots\xi}}{\partial x^\eta} - \Gamma^i_{j\beta\gamma\ldots\xi}\,\Gamma^j_{\alpha\eta} - \Gamma^i_{\alpha j\gamma\ldots\xi}\,\Gamma^j_{\beta\eta} \\ \ldots - \Gamma^i_{\alpha\beta\gamma\ldots j}\,\Gamma^j_{\xi\eta}. \tag{2.4}$$

3. The differential equation (1.1) has a unique set of solutions determined by the initial conditions, $x^i = p^i$ and $dx^i/ds = \xi^i$ when $s = 0$. These solutions may be written in the form:

$$x^i = p^i + \xi^i s - \frac{1}{2!}(\Gamma^i_{\alpha\beta})_P\, \xi^\alpha \xi^\beta s^2 - \frac{1}{3!}(\Gamma^i_{\alpha\beta\gamma})_P\, \xi^\alpha \xi^\beta \xi^\gamma s^3 - \dots \qquad (3.1)$$

in which the subscripts mean that the parentheses are evaluated for $x^i = p^i$, i.e., at the point P.

Let us now substitute $y^i = \xi^i s$ and solve (3.1) for y^i, obtaining

$$y^i = \psi^i(x^1, x^2 \dots, x^n) \qquad (3.2)$$

This determines a transformation of the x's into new coördinates $y^1, y^2 \dots, y^n$. The y's are normal coördinates. They are determined uniquely by the x's, the point P, and the differential equation (1.1). They have the characteristic property that every curve

$$y^i = \xi^i s \qquad (3.3)$$

is a path, i.e., a solution of the equation,

$$\frac{d^2 y^i}{ds^2} + C^i_{\alpha\beta}\frac{dy^\alpha}{ds}\frac{dy^\beta}{ds} = 0 \qquad (3.4)$$

into which (1.1) is transformed by the substitution (3.1). Moreover every path through P is given by (3.3).

4. Substituting (3.3) in (3.4) we obtain

$$C^i_{\alpha\beta}\, \xi^\alpha \xi^\beta = 0 \qquad (4.1)$$

in which the functions $C^i_{\alpha\beta}$ are evaluated for the values of y such that $y^i = \xi^i s$. This is more simply written in the form

$$C^i_{\alpha\beta}\, y^\alpha y^\beta = 0 \qquad (4.2)$$

which is an identity in y^i. Differentiating this with regard to s, we obtain

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} y^\alpha y^\beta + 2 C^i_{\alpha\gamma} y^\alpha\right)\frac{dy^\gamma}{ds} = 0. \qquad (4.21)$$

Since $\frac{dy^\gamma}{ds}$ is arbitrary this gives

$$\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} y^\alpha y^\beta + 2 C^i_{\alpha\gamma} y^\alpha = 0. \qquad (4.22)$$

Multiplying this by y^γ, summing, and using (4.2) we obtain

$$\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} y^\alpha y^\beta y^\gamma = 0. \qquad (4.3)$$

Differentiating once more and repeating the process just described, we obtain

$$\frac{\partial^2 C^i_{\alpha\beta}}{\partial y^\gamma \partial y^\delta} y^\alpha y^\beta y^\gamma y^\delta = 0, \tag{4.4}$$

and in general,

$$\frac{\partial^m C^i_{\alpha\beta}}{\partial y^\gamma \partial y^\delta \ldots \partial y^\xi} y^\alpha y^\beta \ldots y^\xi = 0. \tag{4.m}$$

5. Since the directions ξ^i at the origin are entirely arbitrary it follows from (4.1) that

$$(C^i_{\alpha\beta})_0 = 0. \tag{5.2}$$

If we substitute $y^i = \xi^i s$ in (4.3), divide by s^3, and evaluate for $y^i = 0$ we have

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma}\right)_0 \xi^\alpha \xi^\beta \xi^\gamma = 0.$$

Rewriting this with α,β,γ permuted cyclically and adding, we obtain

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial C^i_{\gamma\alpha}}{\partial y^\beta}\right)_0 \xi^\alpha \xi^\beta \xi^\gamma = 0.$$

Since this form is symmetric and the ξ's are arbitrary,

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial C^i_{\gamma\alpha}}{\partial y^\beta}\right)_0 = 0. \tag{5.3}$$

By a similar argument we obtain from (4.4) the relation:

$$\left(\frac{\partial^2 C^i_{\alpha\beta}}{\partial y^\gamma \partial y^\delta} + \frac{\partial^2 C^i_{\alpha\gamma}}{\partial y^\beta \partial y^\delta} + \frac{\partial^2 C^i_{\alpha\delta}}{\partial y^\beta \partial y^\gamma} + \frac{\partial^2 C^i_{\beta\gamma}}{\partial y^\alpha \partial y^\delta} + \frac{\partial^2 C^i_{\beta\delta}}{\partial y^\alpha \partial y^\gamma} + \frac{\partial^2 C^i_{\gamma\delta}}{\partial y^\alpha \partial y^\beta}\right)_0 = 0. \tag{5.4}$$

By repeating this process we find that *the sum of the* $(m + 2)$ $(m + 1)/2$ *derivatives of the* m^{th} *order of the functions, C, in which any set of* $m + 2$ *integers,* $\alpha,\beta,\ldots,\xi$, $(\leqq m)$ *appear as the subscripts of the C's and the superscripts of the y's, is zero at the origin of normal coördinates.*

6. We now define a sequence of systems of functions of $(x^1,x^2,\ldots,x^n)$, $A^i_{\alpha\beta\ldots\xi}$, by the condition that the value of $A^i_{\alpha\beta\gamma\ldots\xi}$ at any point $(x^1,x^2,\ldots,x^n)$ of the manifold is the value of $\left(\frac{\partial^m C^i_{\alpha\beta}}{\partial y^\gamma \ldots \partial y^\xi}\right)_0$ determined in the system of normal coördinates having $(x^1,x^2,\ldots,x^n)$ as origin.

From (1.2) it follows that

$$A^i_{\alpha\beta\gamma\ldots\xi} = A^i_{\beta\alpha\gamma\ldots\xi}. \tag{6.1}$$

From the fact that all the subscripts of $A^i_{\beta\alpha\gamma\ldots\xi}$ after the first two correspond to differentiation of $C^i_{\alpha\beta}$ it follows that

$$A^i_{\alpha\beta\gamma\ldots\xi} = A^i_{\beta\alpha\delta\ldots\eta}, \tag{6.2}$$

where $\delta\ldots\eta$ stands for any permutation of the subscripts, $\gamma\ldots\xi$. From (5.3) there follows

$$A^i_{\alpha\beta\gamma} + A^i_{\beta\gamma\alpha} + A^i_{\gamma\alpha\beta} = 0. \tag{6.3}$$

This is a special case of an identity,

$$A^i_{\alpha\beta\gamma\ldots\xi} + A^i_{\alpha\gamma\beta\ldots\xi} + \ldots = 0 \tag{6.4}$$

in which there are $m\ (m - 1)/2$ terms, each pair of the m subscripts $\alpha\beta\gamma\ldots\xi$ being the first pair in one and only one term. This identity follows directly from the last theorem of §5.

7. *The system of functions* $A^i_{\alpha\beta\gamma\ldots\xi}$ *with* m *subscripts* ($m \geqq 3$) *is a tensor of order* $m + 1$. This theorem can be inferred directly from the invariantive character of the normal coördinates. But we prefer to prove it here by showing how to express the functions A explicitly in terms of the curvature tensor and its covariant derivatives which have already been proved to be tensors in the paper by Professor Eisenhart and the writer.

8. The curvature tensor is defined by the equation,

$$B^i_{\alpha\beta\gamma} = \left(\frac{\partial\Gamma^i_{\alpha\beta}}{\partial x^\gamma} - \Gamma^i_{j\beta}\,\Gamma^j_{\alpha\gamma}\right) - \left(\frac{\partial\Gamma^i_{\alpha\gamma}}{\partial x^\beta} - \Gamma^i_{j\gamma}\,\Gamma^j_{\alpha\beta} =\right)\Gamma^i_{\alpha\beta\gamma} - \Gamma^i_{\alpha\gamma\beta}. \tag{8.1}$$

(This is the negative of what we denoted by $B^i_{\alpha\beta\gamma}$ in the former paper.) If it is computed in a normal coördinate system and evaluated at the origin, P, of these coördinates, it must satisfy the equations,

$$(B^i_{\alpha\beta\gamma})_P = \left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} - \frac{\partial C^i_{\alpha\gamma}}{\partial y^\beta}\right)_0 \tag{8.2}$$

These equations are equivalent to

$$(2B^i_{\alpha\beta\gamma} + B^i_{\gamma\alpha\beta})_P = 3\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma}\right)_0 - \left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial C^i_{\gamma\alpha}}{\partial y^\beta}\right)_0. \tag{8.3}$$

By means of (5.3) and the definition of $A^i_{\alpha\beta\gamma}$ this leads to

$$A^i_{\alpha\beta\gamma} = \frac{1}{3}\,(2B^i_{\alpha\beta\gamma} + B^i_{\gamma\alpha\beta}). \tag{8.4}$$

By means of the well-known identity,

$$B^i_{\alpha\beta\gamma} + B^i_{\gamma\alpha\beta} + B^i_{\beta\gamma\alpha} = 0, \tag{8.5}$$

this reduces to

$$A^i_{\alpha\beta\gamma} = \frac{1}{3}\,(B^i_{\alpha\beta\gamma} + B^i_{\beta\alpha\gamma}). \tag{8.6}$$

The identity (8.5) is itself a direct consequence of (8.2).

9. In order to extend the formulas of §8 to our sequence of sets of functions $A^i_{\alpha\beta\gamma\ldots\xi}$ we observe that by repeated covariant differentiation of (8.1) we obtain

$$B^i_{\alpha\beta\gamma\delta\ldots\xi} = \frac{\partial^m \Gamma^i_{\alpha\beta}}{\partial x^\gamma \partial x^\delta \ldots \partial x^\xi} - \frac{\partial^m \Gamma^i_{\alpha\gamma}}{\partial x^\beta \partial x^\delta \ldots \partial x^\xi} + \ldots \qquad (9.1)$$

where the term on the left of the equality sign represents an $(m - 3)$rd covariant derivative of $B^i_{\alpha\beta\gamma}$, and the three dots at the right represent terms involving derivatives of the Γ's of order less than m and covariant derivatives of $B^i_{\alpha\beta\gamma}$ of order less than $m - 3$. By writing (9.1) in normal coördinates so that the Γ's in the right member become C's and then evaluating at the origin by means of the definition of the A's, we obtain

$$B^i_{\alpha\beta\gamma\delta\ldots\xi} = A^i_{\alpha\beta\gamma\delta\ldots\xi} - A^i_{\alpha\gamma\beta\delta\ldots\xi} + \ldots \qquad (9.2)$$

in which the three dots represent terms involving A's and B's with fewer than $m - 1$ subscripts.

The equations (9.2) by themselves do not determine the A's uniquely in terms of the B's. When taken in connection with the equations in § 6, however, they can be solved.

10. In order to find this solution we observe that among the A's with a given set of n subscripts any one is determined by its first two subscripts, and these two subscripts are interchangeable (§ 6). We consider the following $m(m - 1)/2$ permutations of the subscripts $(\beta\alpha\gamma\ldots\xi,)$ $(\gamma\beta\alpha\ldots\xi)$, $(\gamma\alpha\delta\ldots\xi)$, $(\delta\gamma\beta\ldots\xi)$, $(\delta\beta\alpha\ldots\xi)$, $(\delta\alpha\epsilon\ldots\xi)$, $(\epsilon\delta\gamma\ldots\xi)$, $(\epsilon\gamma\beta\ldots\xi)\ldots$. The A's with any two successive permutations of this set as subscripts are capable of entering in an equation of the form (9.2). There is thus determined a set of $M - 1$ equations like (9.2), where $M = \frac{1}{2}m(m - 1)$. Let the first of these equations be multiplied by $(M - 1)$, the second by $(M - 2)\ldots$ and the last by 1. On adding the resulting equations we obtain

$$\begin{aligned}(M-1)\,B^i_{\beta\alpha\gamma\ldots\xi} + (M-2)\,B^i_{\gamma\beta\alpha\ldots\xi} + (M-3)\,B^i_{\gamma\alpha\delta\ldots\xi} + \ldots \\ = MA^i_{\beta\alpha\gamma\ldots\xi} - (A^i_{\beta\alpha\gamma\ldots\xi} + A^i_{\gamma\beta\alpha\ldots\xi} + \ldots) + \ldots\end{aligned} \qquad (10.1)$$

The parenthesis of the right member is zero by (6.4). Hence (10.1) reduces to

$$A^i_{\alpha\beta\gamma\ldots\xi} = \frac{1}{M}\left\{(M-1)\,B^i_{\beta\alpha\gamma\ldots\xi} + (M-2)\,B^i_{\gamma\beta\alpha\ldots\xi} + \ldots + \ldots\right\} (10.2)$$

in which the coefficients of the terms in the parenthesis are the integers from 1 to $M - 1$, the permutations of the subscripts of the B's are those indicated in the paragraph above, and the final three dots represent a polynomial in A's and B's with fewer than $(m - 1)$ subscripts. By using (10.2) as a recursion formula this last polynomial is converted into a polynomial in the B's with fewer than $(m - 1)$ subscripts.

This completes the proof that each A is expressible as a polynomial in the B's and determines explicitly the coefficients of the linear expression in the B's with m subscripts which forms a part of this polynomial.

Each of the terms represented by the three dots in (9.2) is evidently a sum of products of A's and B's (such, for example, as $A^i_{\alpha\beta j}\, B^j_{\gamma\delta\epsilon}$) of such a form that it must represent a tensor if the A's and B's which enter into it represent tensors. The same remark follows directly for the terms represented by the three dots in (10.2). Hence by the use of (10.2) as a recursion formula it follows that the A's are all tensors.

11. The rule for determining the permutations of the subscripts of the $M - 1$ functions B which appear linearly in (10.2) can be regarded as a rule for tracing out the points and lines of a configuration analogous to the Desargues Configuration (cf. Veblen and Young, *Projective Geometry*, Vol. I, Chap. 2). To see this it is only necessary to observe that the M permutations of the subscripts of $A^i_{\alpha\beta\gamma\ldots\xi}$ which give functions which are not identical according to (6.1) and (6.2) are in (1 – 1) correspondence with the points of the configuration obtained by taking a plane section of a complete m-point in a projective 3-space.

12. For the case $m = 3$ the formula (10.2) reduces to (8.4). For the case $m = 4$ the terms represented by the three dots in (9.2) are all zero and hence they are all zero in (10.2). Hence the latter formula reduces to

$$A^i_{\alpha\beta\gamma\delta} = \frac{1}{6}\,(5B^i_{\beta\alpha\gamma\delta} + 4\,B^i_{\gamma\beta\alpha\delta} + 3\,B^i_{\gamma\alpha\delta\beta} + 2\,B^i_{\delta\gamma\beta\alpha} + B^i_{\delta\beta\alpha\gamma}) \qquad (11.1)$$

Another formula which follows at once from (9.1) or (11.1) is

$$\begin{aligned} A^i_{\alpha\beta\gamma\delta} = \frac{1}{24}\,[5(B^i_{\alpha\beta\gamma\delta} + B^i_{\beta\alpha\gamma\delta} + B^i_{\alpha\beta\delta\gamma} + B^i_{\beta\alpha\delta\gamma}) \\ - (B^i_{\gamma\delta\alpha\beta} + B^i_{\delta\gamma\alpha\beta} + B^i_{\gamma\delta\beta\alpha} + B^i_{\delta\gamma\beta\alpha})] \end{aligned} \qquad (11.2)$$

From (9.1) there also follows at once the identity of Bianchi,

$$B^i_{\alpha\beta\gamma\delta} + B^i_{\alpha\delta\beta\gamma} + B^i_{\alpha\gamma\delta\beta} = 0, \qquad (11.3)$$

as well as

$$B^i_{\beta\alpha\gamma\delta} + B^i_{\gamma\beta\delta\alpha} + B^i_{\delta\gamma\alpha\beta} + B^i_{\alpha\delta\beta\gamma} = 0 \qquad (11.4)$$

which is one of a sequence of important identities.

PRINCIPAL DIRECTIONS IN A RIEMANNIAN SPACE

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1. A type of principal directions for a Riemannian space of N dimensions has been defined by Ricci (*Atti R. Ist. Veneto,* **63,** p. 1233). Four distinct types are defined in the present paper, of which one is identical with that of Ricci, although defined in a manner somewhat more simple.

We shall adopt the common convention of summation with respect to any index occurring twice in a product, except where the index is a capital letter. The manifold under consideration is of N dimensions; the small Roman indices imply a range or summation from 1 to N, the small Greek indices from 1 to $N-1$. The line element being given by

$$ds^2 = g_{mn} dx_m dx_n,$$

we define in the usual manner

$$G_{mn,st} = \frac{\partial}{\partial x_t}\begin{bmatrix} ms \\ n \end{bmatrix} - \frac{\partial}{\partial x_s}\begin{bmatrix} mt \\ n \end{bmatrix} + g^{ab}\left\{\begin{bmatrix} mt \\ a \end{bmatrix}\begin{bmatrix} ns \\ b \end{bmatrix} - \begin{bmatrix} ms \\ a \end{bmatrix}\begin{bmatrix} nt \\ b \end{bmatrix}\right\} \quad (1.1)$$

$$G_{ns} = g^{mt}\, G_{mn,st} \quad (1.2)$$
$$G = g^{ns}\, G_{ns}.$$

The word "surface" will be used to denote any $(N-1)$-space immersed in the given N-space.

2. Directions defined by invariant relations may be termed *principal.* Any invariant function of direction will, in general, yield such principal directions, corresponding to stationary values of the function. The following are types of principal directions:—

Type I: Consider the family of surfaces, $G =$ constant. Its orthogonal trajectories constitute principal directions; their equations are

$$\frac{\partial G}{\partial x_s} = \theta g_{st}\, \dot{x}_t \qquad (s=1,\ldots, N) \qquad (2.1)$$

where the point denotes differentiation with respect to the arc. Now, for any direction,

$$\dot{G}^2 = \frac{\dfrac{\partial G}{\partial x_s}\dfrac{\partial G}{\partial x_t}\, dx_s\, dx_t}{g_{st}\, dx_s\, dx_t};$$

therefore directions making $\dot{G}^2$ stationary satisfy

$$\frac{\partial G}{\partial x_s}\frac{\partial G}{\partial x_t}\, dx_t = \phi g_{st}\, dx_t \qquad (s = 1, \ldots, N)$$

or

$$\frac{\partial G}{\partial x_s}\dot{G} = \phi g_{st}\,\dot{x}_t \qquad (s = 1, \ldots, N).$$

But these are of the same form as (2.1), writing $\theta\dot{G} = \phi$, and hence the type of principal direction defined above is that for which $\dot{G}^2$, and consequently $\dot{G}$, is stationary.

Type II: Consider a point (P) and a geodesic passing through it. Along this geodesic

$$\ddot{G} = \left[\frac{\partial^2 G}{\partial x_s \partial x_t} - \left\{\begin{matrix} st \\ m \end{matrix}\right\}\frac{\partial G}{\partial x_m}\right]\dot{x}_s\dot{x}_t.$$

This quantity is a function of direction at P, and the principal directions, corresponding to stationary values, are given by

$$\left[\frac{\partial^2 G}{\partial x_s \partial x_t} - \left\{\begin{matrix} st \\ m \end{matrix}\right\}\frac{\partial G}{\partial x_m}\right] dx_t = \theta g_{st}\, dx_t \qquad (s = 1, \ldots, N). \quad (2.2)$$

Type III: The expression $G_{st}\dot{x}_s\dot{x}_t$ is invariant for any given direction. Principal directions, corresponding to stationary values, are given by

$$G_{st}\, dx_t = \theta g_{st} dx_t \qquad (s = 1, \ldots, N). \quad (2.3)$$

Eisenhart (*Proc. N. A. S.*, Vol. 8, No. 2, p. 24) has shown that the principal directions of Ricci (*loc. cit.*) may be expressed in this form. These directions may also be reached from other considerations. Any direction at a point (P) defines a surface consisting of all geodesics passing through P and perpendicular to the given direction. The curvature invariant $(\bar{G})$ of this surface at P depends only on the given direction. Those directions making $\bar{G}$ stationary are principal directions: they may be proved to be identical with those considered above.

Type IV: The expression

$$g^{s_1t_1}\, g^{s_2t_2}\, g^{s_3t_3}\, G_{s_1s_2,s_3s}\, G_{t_1t_2,t_3t}\, \dot{x}_s\dot{x}_t \quad (2.4)$$

is invariant for any given direction. Principal directions, corresponding to stationary values, are given by

$$g^{s_1t_1}\, g^{s_2t_2}\, g^{s_3t_3}\, G_{s_1s_2,s_3s}\, G_{t_1t_2,t_3t}\, dx_t = \theta g_{st}\, dx_t \quad (s = 1, \ldots, N). \quad (2.5)$$

The four types defined above are not intended to be exhaustive of all types of principal directions. Type II, for example, will give principal directions if any other invariant function of position is substituted for G. Let us suppose that the expression (2.4) has the same value for all directions at any given point, but varies from point to point. The principal directions of Type IV are then indeterminate, but (2.4) is an invariant function of position and may therefore be substituted in (2.2) to yield principal directions.

3. In order to prove certain theorems concerning the principal directions defined above, we shall require particular coördinate systems. A system of coördinates will be said to be "O.T.(x_N)" if the parametric lines of x_N are the orthogonal trajectories of the family of surfaces x_N = constant. The necessary and sufficient conditions for an O.T.(x_N) system are easily seen to be

$$g_{N\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1). \qquad (3.1)$$

A special type of O.T.(x_N) system is the "G.O.T.(x_N)" system for which the parametric lines of x_N are geodesics. The equations of a geodesic are

$$\ddot{x}_s + \left\{ {mn \atop s} \right\} \dot{x}_m \dot{x}_n = 0 \qquad (s = 1, \ldots, N) \quad .$$

or

$$g_{st}\ddot{x}_t + \left[{mn \atop s} \right] \dot{x}_m \dot{x}_n = 0 \qquad (s = 1, \ldots, N). \qquad (3.2)$$

The coördinate system being O.T.(x_N), the parametric lines of x_N satisfy (3.2) if, and only if,

$$\left[{NN \atop \sigma} \right] = 0 \qquad (\sigma = 1, \ldots, N-1)$$

and

$$g_{NN}\ddot{x}_N + \left[{NN \atop N} \right] \dot{x}_N^2 = 0.$$

The latter equation is always satisfied for the parametric lines of x_N, by virtue of the equation

$$g_{NN}\dot{x}_N^2 = 1;$$

the former are equivalent to

$$\frac{\partial g_{NN}}{\partial x\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1).$$

Hence we have, as necessary and sufficient conditions for a G.O.T. (x_N) system,

$$g_{N\sigma} = 0, \quad \frac{\partial g_{NN}}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1). \qquad (3.3)$$

If we are given an ∞^1 family of surfaces, we can find an O.T.(x_N) system for which the family is given by x_N = constant. If we are given a single surface and draw the congruence of geodesics normal to it, it follows from the Calculus of Variations that this is a normal congruence, and that any two of the normal surfaces give equal intercepts on all the geodesics. Taking these geodesics as parametric lines of x_N and taking x_N as the distance measured along these geodesics from the given surface, we have a

G.O.T.(x_N) system for which the given surface has the equation $x_N = 0$, and $g_{NN} = 1$ throughout the manifold (cf. Bianchi, Vol. I, p. 336).

4. A "plane" surface—the "superficie geodetiche" of Ricci (*Atti R. Acc. Lincei*, Ser. 5, Vol. 12, p. 409)—may be defined as one whose geodesics are also geodesics of the containing manifold. Let us choose an O.T. (x_N) system for which a certain plane surface has the equation $x_N = 0$. The line element of the surface is given by

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu$$

and the equations of its geodesics are, in the form of (3.2),

$$g_{\sigma\tau}\,\ddot{x}_\tau + \begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix}\dot{x}_\mu\,\dot{x}_\nu = 0 \tag{4.1}$$

$$x_N = 0; \tag{4.2}$$

for such a curve we find, for $\sigma = 1, \ldots, N-1$,

$$\begin{aligned} g_{\sigma t}\,\ddot{x}_t + \begin{bmatrix}mn\\ \sigma\end{bmatrix}\dot{x}_m\,\dot{x}_n &= g_{\sigma\tau}\,\ddot{x}_\tau + \begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix}\dot{x}_\mu\,\dot{x}_\nu, &&\text{by (4.2),}\\ &= 0, &&\text{by (4.1);}\end{aligned}$$

also

$$\begin{aligned} g_{Nt}\,\ddot{x}_t + \begin{bmatrix}mn\\ N\end{bmatrix}\dot{x}_m\,\dot{x}_n &= \begin{bmatrix}\mu\nu\\ N\end{bmatrix}\dot{x}_\mu\,\dot{x}_\nu, &&\text{by (3.1) and (4.2).}\\ &= -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_N}\dot{x}_\mu\,\dot{x}_\nu, &&\text{by (3.1).}\end{aligned}$$

From the definition of a plane surface and by (3.2), this latter quantity must vanish for all arbitrary directions in the surface. Therefore we must have, at all points of the surface,

$$\frac{\partial g_{\mu\nu}}{\partial x_N} = 0 \qquad (\mu,\nu = 1, \ldots, N-1). \tag{4.3}$$

If we are given a plane surface and choose a G.O.T.(x_N) system for which the equation of the surface is $x_N = 0$, and $g_{NN} = 1$, then, at any point of the surface, (3.3) and (4.3) hold. Applying these conditions for the reduction of (1.1) and (1.2), we find that at any point of the surface

$$G_{N\nu,\sigma t} = 0 \qquad (\nu,\sigma,t = 1, \ldots, N-1) \tag{4.4}$$

$$G_{N\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1) \tag{4.5}$$

5. Theorem: *The direction of Type I is contained in those of Type II if, and only if, the lines of Type I are geodesic.* In order to establish this theorem we shall employ an O.T.(x_N) system for which the surface G = constant have the equations x_N = constant. The principal directions of Type I are then given by

$$dx_\sigma = 0 \qquad (\sigma = 1, \ldots, N-1); \tag{5.1}$$

those of Type II by

$$\left.\begin{aligned} &\left[\frac{\partial^2 G}{\partial x_\sigma \partial x_t} - \begin{Bmatrix} \sigma t \\ m \end{Bmatrix} \frac{\partial G}{\partial x_m}\right] dx_t = \theta g_{\sigma t}\, dx_t \quad (\sigma = 1, \ldots, N-1) \\ &\left[\frac{\partial^2 G}{\partial x_N \partial x_t} - \begin{Bmatrix} Nt \\ m \end{Bmatrix} \frac{\partial G}{\partial x_m}\right] dx_t = \theta g_{Nt}\, dx_t. \end{aligned}\right\} \quad (5.2)$$

Applying the conditions,

$$g_{N\sigma} = 0, \qquad \frac{\partial G}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1)$$

to (5.2), we obtain

$$\left.\begin{aligned} -\begin{Bmatrix} \sigma t \\ N \end{Bmatrix} \frac{\partial G}{\partial x_N}\, dx_t &= \theta g_{\sigma\tau}\, dx_\tau \quad (\sigma = 1, \ldots, N-1) \\ \frac{\partial^2 G}{\partial x_N^2}\, dx_N - \begin{Bmatrix} Nt \\ N \end{Bmatrix} \frac{\partial G}{\partial x_N}\, dx_t &= \theta g_{NN}\, dx_N. \end{aligned}\right\} \quad (5.3)$$

It is easily seen that (3.1) imply $g^{N\sigma} = 0$ $(\sigma = 1, \ldots, N-1)$; employing these equations we find

$$\begin{aligned} \begin{Bmatrix} \sigma_\tau \\ N \end{Bmatrix} dx_t &= \frac{1}{2} g^{NN} \frac{\partial g_{NN}}{\partial x_\sigma}\, dx_N - \frac{1}{2} g^{NN} \frac{\partial g_{\sigma\tau}}{\partial x_N}\, dx_\tau \quad (\sigma = 1, \ldots, N-1) \\ \begin{Bmatrix} Nt \\ N \end{Bmatrix} dx_t &= \frac{1}{2} g^{NN} \frac{\partial g_{NN}}{\partial x_N}\, dx_N + \frac{1}{2} g^{NN} \frac{\partial g_{NN}}{\partial x_\tau}\, dx_\tau. \end{aligned}$$

Hence (5.3) become

$$\left.\begin{aligned} &-\frac{1}{2} g^{NN} \frac{\partial G}{\partial x_N} \left(\frac{\partial g_{NN}}{\partial x_\sigma}\, dx_N - \frac{\partial g_{\sigma\tau}}{\partial x_N}\, dx_\tau\right) = \theta g_{\sigma\tau}\, dx_\tau \quad (\sigma = 1, \ldots, N-1) \\ &\left(\frac{\partial^2 G}{\partial x_N^2} - \frac{1}{2} g^{NN} \frac{\partial G}{\partial x_N} \frac{\partial g_{NN}}{\partial x_N}\right) dx_N - \frac{1}{2} g^{NN} \frac{\partial G}{\partial x_N} \frac{\partial g_{NN}}{\partial x_\tau}\, dx_\tau = \theta g_{NN}\, dx_N. \end{aligned}\right\} \quad (5.4)$$

Since we hypothesize that the Type I direction is determinate,

$$\frac{\partial G}{\partial x_N} \neq 0\,;$$

therefore (5.4) are satisfied by (5.1) if, and only if,

$$\frac{\partial g_{NN}}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1).$$

But, the coördinate system being O.T.(x_N), these conditions are necessary and sufficient that the parametric lines of x_N shall be geodesic, by (3.3). Therefore the theorem is established.

6. Theorem: *If there exists a plane surface, its normal direction is a principal direction of both Type III and Type IV.* Let us employ a G.O.T.

(x_N) system for which the equation of the plane surface is $x_N = 0$, and $g_{NN} = 1$. Since (3.3) and (4.5) hold at all points of the surface, the equations (2.3) for the directions of Type III become

$$\left.\begin{aligned} G_{\sigma\tau}\,dx_\tau &= \theta g_{\sigma\tau}\,dx_\tau \quad (\sigma = 1, \ldots, N-1) \\ G_{NN}\,dx_N &= \theta g_{NN}\,dx_N \end{aligned}\right\} \tag{6.1}$$

These equations are satisfied by $dx_\sigma = 0$ $(\sigma = 1, \ldots, N-1)$, which is the direction normal to the plane surface, and thus the theorem is established for Type III. This part of the theorem has also been proved by Ricci (*Atti R. Ist. Ven. loc. cit.*).

By (3.3) the equations (2.5) for the directions of Type IV become

$$g^{s_1t_1}g^{s_2t_2}g^{s_3t_3}G_{s_1s_2,s_3\sigma}G_{t_1t_2,t_3t}\,dx_t = \theta g_{\sigma\tau}dx_\tau \quad (\sigma = 1, \ldots, N-1) \tag{6.2}$$

$$g^{s_1t_1}g^{s_2t_2}g^{s_3t_3}G_{s_1s_2,s_3N}G_{t_1t_2,t_3t}\,dx_t = \theta g_{NN}dx_N. \tag{6.3}$$

Let us consider the surviving terms in the left hand sides of these equations, the relations effecting reductions being from (3.3),

$$g^{N\sigma} = 0 \quad (\sigma = 1, \ldots, N-1);$$

from the well known properties of the tensor-components,

$$G_{NN,st} = 0, \quad G_{st,NN} = 0 \quad (s,t = 1, \ldots, N);$$

while from (4.4) we see that any tensor-component vanishes if one and only one of its indices is N. In (6.2), if $t = N$, then either t_1 or t_2 must be N. Therefore either s_1 or s_2 must be N; therefore s_3 must be N. Hence $t_3 = N$, and the second tensor vanishes. Therefore there are no surviving terms in (6.2) for which $t = N$, and (6.2) may be written

$$A_{\sigma\tau}\,dx_\tau = \theta g_{\sigma\tau}\,dx_\tau \quad (\sigma = 1, \ldots, N-1). \tag{6.4}$$

In (6.3) either s_1 or s_2 must be N. Therefore either t_1 or t_2 must be N; therefore either t_3 or t must be N. But if $t_3 = N$, then $s_3 = N$ and the term vanishes. Therefore $t_3 \pm N$, and the only surviving term is that for which $t = N$. Thus (6.3) may be written

$$B dx_N = \theta g_{NN}\,dx_N. \tag{6.5}$$

But (6.4) and (6.5) are satisfied simultaneously by $dx_\sigma = 0$ $(\sigma = 1, \ldots, N-1)$ and thus the theorem is proved.

In the case $N = 2$, the directions of Types III and IV become indeterminate, since through any point we can draw a "plane surface" (in this case a geodesic curve) so that its normal may have any arbitrarily assigned direction.

PRINCIPAL DIRECTIONS IN THE EINSTEIN SOLAR FIELD

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In the Newtonian manifold of space-time there is at any point one principal direction, namely that for which the three space coördinates are stationary. In the hyperbolic space-time of the Special Relativity Theory, there are no principal directions, the only directions intrinsically definable being those forming the cone

$$-dx^2 - dy^2 - dz^2 + c^2dt^2 = 0.$$

In the previous paper (*Proc. N. A. S.*, p. 198) several types of principal directions have been defined for a Riemannian N-space, but the field equations

$$G_{mn} = 0$$

render indeterminate the principal directions of Types I, II and III as there defined. Eisenhart (*Proc. N. A. S.*, Vol. 8, No. 2, p. 24) has shown that those of Type III are indeterminate for all three forms of Einstein space free from matter. However, it would appear that those of Type IV might exist in the Solar Field; they correspond to stationary values of the invariant function of direction

$$\theta = g^{s_1t_1} g^{s_2t_2} g^{s_3t_3} G_{s_1s_2,s_3s} G_{t_1t_2,t_3t} \frac{dx_s}{ds} \frac{dx_t}{ds}.$$

But, as will be seen, the value of θ at any point proves to be independent of direction and therefore the principal directions of Type IV are indeterminate. However, since θ is an invariant function of position varying from point to point, there will exist principal directions corresponding to stationary values of $d^2\theta/ds^2$ for geodesics drawn in all possible directions. These principal directions are given by

$$\left[\frac{\partial^2\theta}{\partial x_s \partial x_t} - \left\{\begin{matrix} st \\ m \end{matrix}\right\} \frac{\partial\theta}{\partial x_m}\right] dx_t = \phi g_{st}\, dx_t \quad (s = 1, 2, 3, 4) \qquad (1)$$

and are a generalization of Type II.

The manifold under consideration is of four dimensions, and, in accordance with the conventions employed in the foregoing paper small Roman indices imply a range or summation from 1 to 4, small Greek indices from 1 to 3. The line element is given by

$$ds^2 = g_{mn}\, dx_m\, dx_n$$

where

$$g_{11} = -(1-k/x_1)^{-1},\ g_{22} = -x_1^2,\ g_{33} = -x_1^2 sin^2 x_2,$$
$$g_{44} = 1-k/x_1,\ g_{mn} = 0 \quad (m \mp n).$$

Hence we have

$$g^{mn} = \begin{cases} 1/g_{mn} & (m = n) \\ 0 & (m \neq n). \end{cases}$$

Observing that

$$\frac{\partial g_{st}}{\partial x_3} = \frac{\partial g_{st}}{\partial x_4} = 0,$$

we find that any three index symbol, $\begin{bmatrix} mn \\ s \end{bmatrix}$, is zero if just one of the indices is either 3 or 4. Turning to the general expression

$$G_{mn,st} = \frac{\partial}{\partial x_t}\begin{bmatrix} ms \\ n \end{bmatrix} - \frac{\partial}{\partial x_s}\begin{bmatrix} mt \\ n \end{bmatrix} + g^{ab}\left\{\begin{bmatrix} mt \\ a \end{bmatrix}\begin{bmatrix} ns \\ b \end{bmatrix} - \begin{bmatrix} ms \\ a \end{bmatrix}\begin{bmatrix} nt \\ b \end{bmatrix}\right\} \tag{2}$$

we find

$$G_{4\nu,\sigma\tau} = g^{ab}\left\{\begin{bmatrix} 4\tau \\ a \end{bmatrix}\begin{bmatrix} \nu\sigma \\ b \end{bmatrix} - \begin{bmatrix} 4\sigma \\ a \end{bmatrix}\begin{bmatrix} \nu\tau \\ b \end{bmatrix}\right\}.$$

Now $\begin{bmatrix} 4\tau \\ a \end{bmatrix}$ vanishes unless $a = 4$, $\begin{bmatrix} \nu\sigma \\ b \end{bmatrix}$ vanishes if $b = 4$, while g^{ab} vanishes unless $a = b$. Therefore

$$g^{ab}\begin{bmatrix} 4\tau \\ a \end{bmatrix}\begin{bmatrix} \nu\sigma \\ b \end{bmatrix} = 0;$$

applying similar reasoning to the second part of the expression, we find

$$G_{4\nu,\sigma\tau} = 0.$$

Thus any tensor-component with just one index equal to 4 vanishes. Similarly any tensor-component with just one index equal to 3 vanishes.

From (2) we find

$$G_{4\nu,\sigma 4} = \frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_\nu \partial x_\sigma} - \frac{1}{2}g^{11}\begin{bmatrix} \nu\sigma \\ 1 \end{bmatrix}\frac{\partial g_{44}}{\partial x_1} - \frac{1}{4}g^{44}\frac{\partial g_{44}}{\partial x_\nu}\frac{\partial g_{44}}{\partial x_\sigma}, \tag{3}$$

and hence

$$G_{4\nu,\sigma 4} = 0 \quad (\nu \neq \sigma).$$

The surviving members of the class given in (3) are

$$\left.\begin{aligned} G_{41,14} &= \frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_1^2} - \frac{1}{4}g^{11}\frac{\partial g_{11}}{\partial x_1}\frac{\partial g_{44}}{\partial x_1} - \frac{1}{4}g^{44}\left(\frac{\partial g_{44}}{\partial x_1}\right)^2 &&= -\frac{k}{x_1^3} \\ G_{42,24} &= \frac{1}{4}g^{11}\frac{\partial g_{22}}{\partial x_1}\frac{\partial g_{44}}{\partial x_1} &&= \frac{1}{2}\frac{k}{x_1}\left(1 - \frac{k}{x_1}\right) \\ G_{43,34} &= \frac{1}{4}g^{11}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{44}}{\partial x_1} &&= \frac{1}{2}\frac{k}{x_1}\sin^2 x_2\left(1 - \frac{k}{x_1}\right) \end{aligned}\right\} \tag{4}$$

From (2) we find

$$G_{\mu\nu,\sigma\tau} = \frac{\partial}{\partial x_\tau}\begin{bmatrix}\mu\sigma\\ \nu\end{bmatrix} - \frac{\partial}{\partial x_\sigma}\begin{bmatrix}\mu\tau\\ \nu\end{bmatrix} + g^{\alpha\beta}\left\{\begin{bmatrix}\mu\tau\\ \alpha\end{bmatrix}\begin{bmatrix}\nu\sigma\\ \beta\end{bmatrix} - \begin{bmatrix}\mu\sigma\\ \alpha\end{bmatrix}\begin{bmatrix}\nu\tau\\ \beta\end{bmatrix}\right\}. \qquad (5)$$

Since the presence of just one 3 among the indices makes the tensor-component vanish, the surviving independent members of this class are

$$G_{31,13}, \quad G_{31,23}, \quad G_{32,23}, \quad G_{21,12};$$

we calculate them from (5):—

$$G_{31,13} = \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x_1^2} - \frac{1}{4}g^{11}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{11}}{\partial x_1} - \frac{1}{4}g^{33}\left(\frac{\partial g_{33}}{\partial x_1}\right)^2 = -\frac{1}{2}\sin^2 x_2\,\frac{k/x_1}{1-k/x_1} \qquad (6)$$

$$\left.\begin{aligned} G_{31,23} &= \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x_1 \partial x_2} - \frac{1}{4}g^{22}\frac{\partial g_{33}}{\partial x_2}\frac{\partial g_{22}}{\partial x_1} - \frac{1}{4}g^{33}\frac{\partial g_{33}}{\partial x_2}\frac{\partial g_{33}}{\partial x_1} = 0 \\ G_{32,23} &= \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x_2^2} + \frac{1}{4}g^{11}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{22}}{\partial x_1} - \frac{1}{4}g^{33}\left(\frac{\partial g_{33}}{\partial x_2}\right)^2 = kx_1\sin^2 x_2 \\ G_{21,12} &= \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x_1^2} - \frac{1}{4}g^{11}\frac{\partial g_{22}}{\partial x_1}\frac{\partial g_{11}}{\partial x_1} - \frac{1}{4}g^{22}\left(\frac{\partial g_{22}}{\partial x_1}\right)^2 = -\frac{1}{2}\frac{k/x_1}{1-k/x_1} \end{aligned}\right\} \qquad (7)$$

The complete list of surviving components, derivable from (4), (6) and (7), is as follows:—

$$\text{of type } G_{s_1s_2,s_31} \quad \left\{\begin{matrix} G_{21,21} & G_{12,21} \\ G_{31,31} & G_{13,31} \\ G_{41,41} & G_{14,41}\end{matrix}\right. ; \qquad \text{of type } G_{s_1s_2,s_32} \quad \left\{\begin{matrix} G_{12,12} & G_{21,12} \\ G_{32,32} & G_{23,32} \\ G_{42,42} & G_{24,42}\end{matrix}\right. ;$$

$$\text{of type } G_{s_1s_2,s_33} \quad \left\{\begin{matrix} G_{13,13} & G_{31,13} \\ G_{23,23} & G_{32,23} \\ G_{43,43} & G_{34,43}\end{matrix}\right. ; \qquad \text{of type } G_{s_1s_2,s_34} \quad \left\{\begin{matrix} G_{14,14} & G_{41,14} \\ G_{24,24} & G_{42,24} \\ G_{34,34} & G_{43,34}\end{matrix}\right. .$$

The equations of the principal directions of Type IV are

$$\theta g_{st}\,dx_t = g^{s_1t_1}g^{s_2t_2}g^{s_3t_3}G_{s_1s_2,s_3s}G_{t_1t_2,t_3t}\,dx_t \quad (s = 1, 2, 3, 4); \qquad (8)$$

these become

$$\begin{aligned} &\text{for } s = 1, && \theta g_{11}dx_1 = 2g^{11}\left[(g^{22}G_{21,12})^2 + (g^{33}G_{31,13})^2 + (g^{44}G_{41,14})^2\right]dx_1; \\ &\text{for } s = 2, && \theta g_{22}dx_2 = 2g^{22}\left[(g^{11}G_{12,21})^2 + (g^{33}G_{32,23})^2 + (g^{44}G_{42,24})^2\right]dx_2; \\ &\text{for } s = 3, && \theta g_{33}dx_3 = 2g^{33}\left[(g^{11}G_{13,31})^2 + (g^{22}G_{23,32})^2 + (g^{44}G_{43,34})^2\right]dx_3; \\ &\text{for } s = 4, && \theta g_{44}dx_4 = 2g^{44}\left[(g^{11}G_{14,41})^2 + (g^{22}G_{24,42})^2 + (g^{33}G_{34,43})^2\right]dx_4. \end{aligned}$$

On substitution we obtain

$$\theta dx_1 = 3\frac{k^2}{x_1^6}dx_1, \qquad \theta dx_2 = 3\frac{k^2}{x_1^6}dx_2,$$

$$\theta dx_3 = 3\frac{k^2}{x_1^6}dx_3, \qquad \theta dx_4 = 3\frac{k^2}{x_1^6}dx_4.$$

Thus the principal directions of Type IV are indeterminate, and (8) define an invariant function of position

$$\theta = g^{s_1t_1} g^{s_2t_2} g^{s_3t_3} G_{s_1s_2,s_3s} G_{t_1t_2,t_3t} \frac{dx_s}{ds}\frac{dx_t}{ds} = 3\frac{k^2}{x_1^6}.$$

Substituting this value for θ in (1), we obtain, after reduction, the following equations for principal directions:—

$$\phi dx_1 = 9\frac{k^2}{x_1^8}\left(15\frac{k}{x_1} - 14\right) dx_1, \qquad \phi dx_2 = 18\frac{k^2}{x_1^8}\left(1 - \frac{k}{x_1}\right) dx_2,$$

$$\phi dx_3 = 18\frac{k^2}{x_1^8}\left(1 - \frac{k}{x_1}\right) dx_3, \qquad \phi dx_4 = 9\frac{k^3}{x_1^9} dx_4.$$

These equations determine the following directions:—

(i) the parametric lines of x_1, $(dx_2 = dx_3 = dx_4 = 0)$;

(ii) any direction making $dx_1 = dx_4 = 0$;

(iii) the parametric lines of x_4, $(dx_1 = dx_2 = dx_3 = 0)$.

It might be said that these principal directions illustrate both the radial and the stationary characters of the field.

FIELDS OF PARALLEL VECTORS IN THE GEOMETRY OF PATHS

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1. In a former-paper (these PROCEEDINGS, Feb. 1922) Professor Veblen and the writer considered the geometry of a general space from the point of view of the paths in such a space—the paths being a generalization of straight lines in euclidean space. From this point of view it is natural to think of the tangents to a path as being parallel to one another. In this way our ideas may be coördinated with those of Weyl and Eddington who have considered parallelism to be fundamental rather than the paths which we so consider. It is the purpose of this note to determine the geometries which possess one or more fields of parallel vectors, which accordingly define a significant direction, or directions, at each point of the space.

2. The equations of the paths are taken in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \qquad (2.1)$$

where x^i $(i = 1, \ldots n)$ are the coördinates of a point of a path expressed as functions of a parameter s; $\Gamma^i_{\alpha\beta}$ are functions of the x's such that $\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$.

The components dx^i/ds of the vector tangent to a path being contravariant we put $dx^i/ds = A^i$. In the former paper we observed that the theory of covariant differentiation can be generalized to the geometry of paths by replacing the Christoffel symbols $\left\{ {jk \atop i} \right\}$ by Γ^i_{jk}. Thus the quantities

$$A^i_j = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{\alpha j} A^\alpha \tag{2.2}$$

are the covariant derivatives of A^i; they are the components of a mixed tensor of the second order. Thus $A^i_j\, dx^j$ expresses in invariant form the first variation of the components of A^i as the x's vary. Hence if we write (2.1) in the form

$$A^i_j\, dx^j/ds = 0, \tag{2.3}$$

we see that the first variation of the components of the tangent vector to a path is equal to zero. In this sense we speak of the tangents to a path as parallel.

Suppose now that A^i are the components of any contravariant vector whatever, and consider the vectors at points of any curve C not necessarily a path. The components A^i and coördinates x^i along C are expressible in terms of a parameter s, and dx^i/ds are the components of the tangent to C. If these functions are such that equations (2.3) are satisfied, we say that the vectors A^i are *parallel to one another with respect to the curve.* In particular the tangents to a path are parallel with respect to it. Some time ago Professor Veblen, in discussing the covariant derivative of a tensor, pointed out that it should be interpreted as the system of turning components of the given tensor with respect to the given direction. In this sense (2.3) expresses the fact that the turning components of the vector along the curve are zero.

In order that our definition may be such that if A^i are the components of parallel vectors with respect to a curve so also are φA^i, where φ is a scalar, we say that the vectors of components A^i are parallel with respect to a curve whose tangents have the components dx^j/ds, provided that a scalar function φ exists such that

$$\left(\frac{\partial A^i}{\partial x^j} + \Gamma^i_{\alpha j} A^\alpha - \frac{\partial \log \varphi}{\partial x^j} A^i \right) \frac{dx^j}{ds} = 0. \tag{2.4}$$

3. Equation (2.4) is satisfied independently of the curve, if

$$\frac{\partial A^i}{\partial x^j} + \Gamma^i_{\alpha j} A^\alpha - \frac{\partial \log \varphi}{\partial x^j} A^i = 0, \tag{3.1}$$

that is

$$A^i_j - \frac{\partial \log \varphi}{\partial x^j} A^i = 0. \tag{3.2}$$

Consequently whenever there exists a field of vectors satisfying (3.1), all the vectors are parallel to one another for any curve, and thus there is a significant direction at each point of the space.

For the second covariant derivatives of any contravariant vector A^i we have the identity

$$A^i_{jk} - A^i_{kj} = - A^\alpha B^i_{\alpha jk}, \tag{3.3}$$

where

$$B^i_{\alpha jk} = \frac{\partial \Gamma^i_{\alpha k}}{\partial x^j} - \frac{\partial \Gamma^i_{\alpha j}}{\partial x^k} + \Gamma^\beta_{\alpha k} \Gamma^i_{\beta j} - \Gamma^\beta_{\alpha j} \Gamma^i_{\beta k}, \tag{3.4}$$

that is the B's are the components of the curvature tensor, as defined in the former paper. From (3.3) it follows that the conditions of integrability of (3.1) are

$$A^\alpha B^i_{\alpha jk} = 0. \tag{3.5}$$

From this equation it follows that a necessary and sufficient condition that (3.1) be completely integrable, that is that there exists a field of vectors parallel to any given vector is $B^i_{\alpha jk} = 0$. From the results of the former paper it follows that in this case the space is euclidean.

If the space is not euclidean, a necessary condition that the A's given by (3.5) shall satisfy (3.2) is

$$A^\alpha B^i_{\alpha jkl} = 0, \tag{3.6}$$

where $B^i_{\alpha jkl}$ is the covariant derivative of $B^i_{\alpha jk}$.

Suppose now that the rank of the matrix of equations (3.5) is such that these equations admit a set of solutions A^i determined to within a scalar factor, and that these functions satisfy (3.6). Differentiating (3.5) covariantly with respect to x^l and taking account of (3.6), we have that the functions A^α_l must satisfy (3.5). In consequence of the above assumptions, it follows that

$$A^\alpha_l = A^\alpha \varphi_l \tag{3.7}$$

where φ_l is a covariant vector. Substituting in (3.3) and making use of (3.5), we find that φ_l is a gradient, and consequently (3.7) is of the form (3.2).

The case when equations (3.5) admit m $(<n)$ sets of solutions, in terms of which any set of solutions is linearly expressible can be handled by a method similar to that used in §7 of the former paper. In this case any vector at a point P in the m-fold bundle of vectors determined by the m vectors at P is parallel to a vector in the corresponding bundle at any other point of the space.

4. In the preceding section we have given the conditions for one or more fields of vectors in invariantive form. Now we shall show how all such fields may be obtained by making a suitable choice of coördinates.

Suppose we have m $(<n)$ fields of parallel vectors of components $A^i_{(p)}$, where $i = 1, \text{-} \text{-} \text{-} n;\ p = 1, \text{-} \text{-} \text{-} m$. If $B^i_{(p)}$ denote the components in another set of variables y^i, we have

$$B^i_{(p)} = A^\alpha_{(p)} \frac{\partial y^i}{\partial x^\alpha}. \tag{4.1}$$

If we show that there exist n independent functions y^i such that

$$X_p\,(y^i) \equiv A^\alpha_{(p)} \frac{\partial y^i}{\partial x^\alpha} = 0 \quad (i \neq p), \tag{4.2}$$

then in the new set of coördinates all the components of the vectors will be zero except those of the form $B^p_{(p)}$.

If we form the Poisson operator for (4.2), namely $(X_pX_q - X_qX_p)(y^i)$, we have in consequence of (3.1)

$$(X_pX_q - X_qX_p)\,(y^i) = A^\alpha_{(p)} \frac{\partial \log \varphi_q}{\partial x^\alpha} X_q\,(y^i) - A^\alpha_{(q)} \frac{\partial \log \varphi_p}{\partial x^\alpha} X_p\,(y^i),$$

where φ_p $(p = 1, \ldots m)$ are the functions φ appearing in equations of the form (3.1). Hence (4.2) is a complete system.

If we let p take all the values from 1 to m, there are in accordance with the theory of complete systems $n - m$ independent solutions, which we take for $y^{m+1}, \ldots, y^n$. If we exclude from the system (4.2) the equation $X_r = 0$ where r has a value from 1 to m, we have a complete system of $m - 1$ equations, of which $n - m$ independent solutions are $y^{m+1}, \ldots, \ldots y^n$, and the other we take for y^r. Hence if there exist m fields of parallel vectors, the coördinates can be chosen so that all the components are zero except those of theform $A^p_{(p)} (p = 1, \ldots, m)$, and consequently our problem reduces to the determination of geometries for which equations (3.1) admit solutions of this kind.

5. In order that equations (3.1) admit solutions $A^1_{(1)} \neq 0$, $A^i_{(1)} = 0$ $(i \neq 1)$, we must have

$$\Gamma^1_{1j} = 0, \quad \Gamma^i_{1j} = \frac{\partial}{\partial x^j} \log \psi \quad \begin{pmatrix} i = 2, \ldots n \\ j = 1, \ldots n \end{pmatrix}, \tag{5.1}$$

where ψ is an arbitrary function of the x's, and then $A^1_{(1)} = \varphi/\psi$. Consequently if we choose the Γ's with one or two subscripts 1 as given by (5.1), and take the others as arbitrary functions of the x's, we have the most general geometry with one field of parallel contravariant vectors.

In like manner any geometry with m fields of parallel contravariant vectors can be obtained by choosing

$$\Gamma^i_{pj} = 0 \quad \Gamma^p_{pj} = \frac{\partial}{\partial x^j} \log \psi_p \quad \begin{pmatrix} p = 1, \ldots m \\ i \neq p \\ j = 1, \ldots n \end{pmatrix}, \tag{5.2}$$

where ψ_p is an arbitrary function of $x^p, x^{m+1}, \ldots, x^n$.

6. Suppose now that the geometry is Riemannian, the fundamental form being

$$ds^2 = g_{ij}\,dx^i\,dx^j \quad (g_{ij} = g_{ji}) \tag{6.1}$$

Since

$$\Gamma^i_{1j} = \frac{1}{2} g^{i\alpha}\left(\frac{\partial g_{1\alpha}}{\partial x^j} + \frac{\partial g_{\alpha j}}{\partial x^1} - \frac{\partial g_{1j}}{\partial x^\alpha}\right),$$

equations (5.1) are in this case equivalent to

$$\frac{\partial g_{1\alpha}}{\partial x^j} + \frac{\partial g_{\alpha j}}{\partial x^1} - \frac{\partial g_{1j}}{\partial x^\alpha} = 2g_{1\alpha}\frac{\partial}{\partial x^j}\log\psi \tag{6.2}$$

for $\alpha, j = 1, \ldots n$. When we take $\alpha = j = 1$, we find $g_{11} = \psi^2\Phi(x^2, \ldots x^n)$. When we take $j = 1$, $\alpha \neq 1$, we find

$$\frac{\partial}{\partial x^1}\left(\frac{g_{1\alpha}}{\psi}\right) = \Phi\frac{\partial\psi}{\partial x^\alpha} + \frac{\psi}{2}\frac{\partial\Phi}{\partial x^\alpha}. \tag{6.3}$$

Consequently $g_{1\alpha}$ are given by quadratures, and likewise $g_{\alpha j}$ ($\alpha \neq 1$, $j \neq 1$)from (6.2).

In particular if ψ = const., by interchanging α and j in (6.2), we find that $\partial g_{\alpha j}/\partial x^1 = 0$ for all values of α and j. Moreover (6.2) becomes

$$\frac{\partial g_{1\alpha}}{\partial x^j} = \frac{\partial g_{1j}}{\partial x^\alpha}. \tag{6.4}$$

As a first consequence of this equation we have that g_{11} is a constant which may be taken equal to unity. Again (6.4) are the necessary and sufficient conditions that

$$g_{1\alpha}\,dx^\alpha = dx^1 + d\varphi\,(x^2, \ldots, x^n).$$

If then x^1 is replaced by $x^1 - \varphi\,(x^2, \ldots x^n)$, the form (6.1) becomes

$$ds^2 = dx_1^2 + g_{ij}\,dx^i\,dx^j \quad (i, j = 2, \ldots n), \tag{6.5}$$

where g_{ij} are independent of x^1. A space with linear element (6.5) is the most general which admits a translation into itself. (Bianchi, *Teoria dei gruppi continui*, Pisa, 1918, p. 500.) The space-time manifold of four dimensions used by Einstein in his cosmological considerations is of the type (6.5), x^1 being the coördinate of time.

In the case of $m(<n)$ fields of parallel vectors for which $\Gamma^i_{pj} = 0$, ($p = 1, \ldots m$; $i, j = 1, \ldots, n$), these equations are equivalent to

$$\frac{\partial g_{ji}}{\partial x^p} = 0, \quad \frac{\partial g_{pi}}{\partial x^j} = \frac{\partial g_{pj}}{\partial x^i}. \tag{6.6}$$

From the first of these it follows that all of the functions g_{ij} are independent of $x^1, \ldots x^m$. From the second of (6.6) we find that g_{pq} ($pq = 1,$

..m) are constants. Then as in the above case we show that the linear element can be put in the form

$$ds^2 = (dx^1)^2 + \,..\, + (dx^m)^2 + g_{ij}\, dx^i\, dx^j \quad (i, j = m+1, \,..\, n),$$

where g_{ij} are independent of $x^1, \ldots, x^m$. When $m = n - 1$, equation (6.7) is reducible to the euclidean form.

7. If A^i are the contravariant components of a vector in a Riemannian geometry, its covariant components A_i, are given by

$$A^i = g^{i\alpha} A_\alpha. \tag{7.1}$$

When this expression is substituted in (3.1), we obtain

$$\frac{\partial A_i}{\partial x^j} - \Gamma^\alpha_{ij} A_\alpha - A_i \frac{\partial \log \varphi}{\partial x^j} = 0. \tag{7.2}$$

When we are dealing with a non-Riemannian geometry we say that a field of parallel covariant vectors is one which satisfies (7.2).

The conditions of integrability of (7.2) are

$$A_\alpha B^\alpha_{ijk} = 0. \tag{7.3}$$

In order that the A's given by (7.3) shall satisfy (7.2) it is necessary that

$$A_\alpha B^\alpha_{ijkl} = 0. \tag{7.4}$$

As in the case of contravariant vectors, it can be shown that when there are m ($< n$) independent sets of solutions of (7.3) which satisfy (7.4) there exist m fields of parallel, covariant vectors.

The methods of §§ 4, 5 cannot be applied to the case of covariant fields.

ON THE INFLUENCE OF DENSITY OF POPULATION UPON THE RATE OF REPRODUCTION IN DROSOPHILA[1]

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It has long been known that degree of crowding of organisms in a given space, or the density of the population, has an influence upon various vital processes of the individuals composing the population. In the matter of growth Semper[2] and before him Jabez Hogg[3] showed that volume of water apart from food and other conditions has an influence upon the rate. This subject has again been studied recently by Bilski[4]. Farr[5] showed that there is in man a definite relation between density of population and the death rate. This old work of Farr's has recently been gone over carefully and confirmed by Brownlee.[6] Drzwina and Bohn[7] show that a particular concentration of a toxic substance, just lethal for a single individual in a given volume of water (working with such organisms as infusoria, planarians, hydra, tadpoles, etc.), will be sub-lethal if several individuals are present in the same fixed volume of water.

Without undertaking at this time any systematic review of the literature on this subject it is our purpose in this paper to present some experimental data on the influence of density of population upon a different biological function than those mentioned above, namely specific reproductive rate. This variable we define as measured by the number of adults (imagoes) produced per mated female per day over a specified period. Our interest in this problem arose originally from purely practical considerations. Anyone doing experimental work with *Drosophila* has constantly arising this situation: he wishes to start an experiment with a large number of flies of homogeneous make-up as a group. What is the most expeditious way to get the desired numbers while still maintaining the requirement of homogeneity? Is it better to mate a considerable number of pairs in one bottle, to furnish the desired progeny, and if so, how many pairs? Or is it better to mate up one or two pairs only in each of several bottles? Which plan will give the largest progeny group? Not being able to answer this practically important question satisfactorily either by *a priori* reasoning, or by perusal of the *Drosophila* literature, it was decided to test it experimentally. The results proved to have a considerably wider interest than that implied in the purely practical question.

Plan of Experiment.—Mass matings were made on March 21, 1921 of flies from our line 107, the characteristics of which have been described by Pearl and Parker.[8] When the offspring from the matings emerged they were used to make up the matings of this experiment, according to the following plan. Half pint milk bottles were used, and the food arrangements, etc., were according to the standard *Drosophila* technique of this laboratory, fully described by Pearl and Parker.[9] The composition of the matings was as follows:

SERIES	NO. OF BOTTLES	MATED PAIRS IN EACH BOTTLE AT START
1	4	1
2	4	2
3	4	3
4	4	4
5	4	5
6	4	6
7	4	7
8	4	8
9	4	9
10	3	10
11	3	12
12	3	15
13	3	20
14	1	25
15	3	30
16	2	50

These bottles were put into a 25° incubator. At the end of 8 days the parent flies still surviving were transferred to fresh half pint bottles and allowed to continue breeding for a second period of 8 days.

All the progeny flies as they emerged as imagoes from the two series of bottles were counted and recorded as to sex. No account was kept of larvae or pupae. We are dealing here with net reproductive capacity, just as in certain of our poultry work (Pearl[10]). Food and temperature conditions were constant for all bottles throughout the experiment. The only significant variable between the several series was the density of population. There occurred various accidents to some of the bottles in the course of the experiment. Some of the mated flies died; food occasionally rose in the bottles and killed a few individuals, etc. In reducing the results to a per ♀ day basis allowance has been made for these accidents, and the final results set forth below represent the outcome for normal mated flies living under normal conditions.

Inasmuch as only 16 day's progeny was counted the results cannot be directly stated in terms of total reproductive capacity. Instead they have been tabulated as the rate of reproduction per ♀ day over a 16 day period after emergency. It is quite certain however that the absolute number of progeny per ♀ in the first 16 days of life is highly correlated with the unknown total reproductive capacity. This we feel sure of from many observations in *Drosophila* and from analogous observations on fecundity in the domestic fowl.

Results.—The results of the experiments are exhibited in Table 1, and graphically in figure 1. The table contains the following data: (*a*) the numbers of mated pairs per bottle at the start. The numbers of mated flies per bottle were therefore twice these figures at the start; (*b*) the mean population of mated flies over the whole period of 16 days. If no accidents had occurred these figures would have been 2, 4, 6, 8, 10, 12, 14, 16, etc. But accidents and deaths did occur. The figures set down represent the actual average number of flies present in each bottle over the period of the experiments, taken as a whole. (*c*) The mean number of flies per cubic centimeter of free air space in the bottle. Since all the bottles were by careful measurement of the food made to contain the same number of cc. (130) of free air space, it follows that the entries in this column are simply a constant ($^1/_{130}$) times the entries in the preceding column. They are inserted merely to show what the actual density in these experiments was. (*d*) The mean number of flies per square centimeter of agar, growing yeast on its surface, in each bottle. The same remarks apply to this as to the cubic density figure. It corresponds exactly to the density of population figure used in human vital statistics. The food area per bottle was 23.76 cm.^{2} (*e*) The total female days, meaning the number got by adding together for all females the number of days that each was in the experiment. Thus if a female lived throughout the experiment she contributed 16 ♀ days (1×16) to the total; if she lived 7 days she contributed 7 ♀ days (1×7) and so on. (*f*) Total absolute number of imagoes produced in 16 days. These absolute figures vary of course with the accidental circumstances affecting the mated population. (*g*) The observed rate of reproduction, given as number of offspring imagoes per mated female per day. (*h*) The theoretical rate of reproduction (number of imagoes per mated female per day) as calculated from equation (*i*) below.

TABLE 1

EXPERIMENTAL DATA

PAIRS PER BOTTLE AT START	MEAN POP. AVERAGE 16 DAYS	MEAN FLIES PER CM³ OF SPACE	MEAN FLIES PER CM² OF FOOD AREA	TOTAL ♀ DAYS	TOTAL IMAGOES IN 16 DAY	IMAGOES PER MATED♀ PER DAY	y FROM EQUATION (i)
1	2.00	.015	.084	63	1348	21.4	21.1
2	3.40	.026	.143	89	1124	12.6	14.5
3	5.50	.042	.231	173.5	1877	10.8	10.2
4	7.32	.056	.308	232	2100	9.1	8.1
5	9.29	.071	.391	267	1585	5.9	6.7
6	10.29	.079	.433	298.5	2034	6.8	6.2
7	11.51	.089	.484	377.5	1951	5.2	5.6
8	14.48	.111	.609	436.5	1996	4.6	4.6
9	16.62	.128	.700	519	2117	4.1	4.0
10	18.74	.144	.789	275	855	3.1	3.6
12	21.70	.167	.913	504	1746	3.5	3.1
15	26.02	.200	1.095	609.5	1389	2.3	2.5
20	34.02	.262	1.432	754	1666	2.2	1.8
25	47.75	.367	2.010	384	388	1.0	1.1
30	47.73	.367	2.009	1069.5	1272	1.2	1.1
50	90.66	.697	3.816	1446.5	474	.33	.34

It is at once apparent from this table that there is a profound and regular change in the rate of reproduction of *Drosophila*, under the conditions of these experiments, with increasing density of population. The rate of reproduction per mated female per day declines as density of population increases, at first extremely rapidly and then more and more slowly at higher densities. The total number of progeny flies (23,922) is great enough to give confidence in the results.

A better appreciation may be obtained of the nature and regularity of this change if we put the results in graphic form. This is done in Fig. 1. The smooth curve passing through the circles is the graph of equation (i) fitted to the observations by the method of least squares. In this equation y denotes imagoes per mated female per day, and x denotes mean density of the mated population (measured as flies per bottle) over the whole 16 day period.

$$y = 34.53\, e^{-.018x} \quad x^{-.658} \qquad \text{(i)}$$

Or, in logarithmic form,

$$\log y = 1.54 - .008\, x - .658 \log x \qquad \text{(ii)}$$

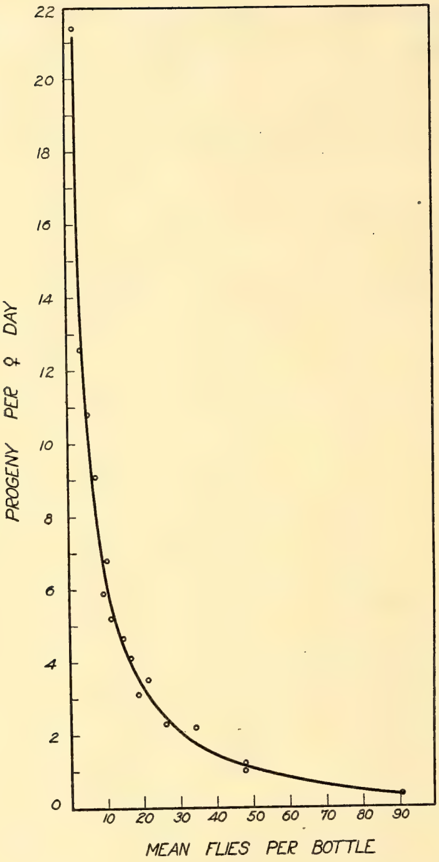

FIG. 1

Showing changes in rate of reproduction of *Drosophila* with increasing density of the mated population. The circles give the observations and the smooth curve is thegraph of the logarithmic equation discussed in the text.

It is at once apparent that this equation describes the observed facts with extraordinary precision. Rarely, even in a physical or chemical experiment, does one get such close agreement as is here shown between observation and theory. Plainly the curve is the expression of the law relating these two phenomena, rate of reproduction and density of population.

It is of interest to note that this equation is essentially identical in form with Farr's Law relating death-rate to population. Farr's original equation was the following, put in logarithmic form:

$$\log D = \log a + k \log d$$

where D denotes death-rate and d density of population and a and k are constants. Brownlee has slightly modified the constants but the essential form remains the same. Now in our equation (ii) the term in x is obviously of no practical importance, and could be left out entirely without visibly affecting the fit. It was only put in in a spirit of meticulous curve-fitting accuracy. With this term out our equation becomes in form identical with Farr's except that we have shifted the origin. The only essential difference is that in his case death-rate is shown to vary directly with density of population, while in the present case rate of reproduction varies inversely with density, but according to the same law. Death and reproduction are both fundamental and antithetical biological phenomena. What we know about the matter suggests that density of population has a profound effect upon many if not all general and fundamental biological processes. Brownlee points out that the same law holds in racing for the relationship between the record time for any definite length of course and the length itself, and for other biological and biochemical phenomena.

Bilski (*loc. cit.*) uses essentially the same equation to describe the influence of density of population and growth in tadpoles. Certainly the subject deserves further study. Possibly our present results offer a clue to a part at least of the solution of the problem of the practically world-wide decline in the birth-rate which had been going on for more than a quarter of a century before the outbreak of the war.

Discussion.—How can the result be explained? The extreme closeness of the observations to the theoretical curve indicates clearly that we are dealing here with no haphazard performance, but with a highly lawful phenomenon. The first point that occurs to one is that since we are dealing with imagoes only, the result may perhaps be explained by larval and pupal elimination due to overcrowding in these stages. In the bottle say with 20 females there were laid, on this view, something like 20 times as many eggs as in the bottle containing one female, these eggs developed into 20 times as many larvae, there was not room or food for so many larvae, therefore many of them were selectively eliminated, and the survivors which got through to the imago stage indicated only a low reproduction rate, which represented however only the capacity of 80 cc. of banana-agar or 23.76 cm.^{2} of food surface to nourish larvae, and nothing else.

Plausible as this hypothesis is—the more so because it would at once indicate why our rate of reproduction variable follows Farr's Law, since it would mean Farr's Law applying to the larvae—it cannot possibly account for the facts. To demonstrate this we have only to examine with some care the absolute figures given in the fifth column of Table 1. In the first place let it be recalled that the greatest drop in rate of reproduction (imagoes per ♀ per day) occurs as we pass from a mean density of 2 flies per bottle to one of 3.4 flies per bottle (1 and 2 pairs originally). The next greatest drop in the rate is from the bottles of density 3.40 to those of density 5.50. Now at all these low densities one in fact cannot possibly speak of larval crowding at all. This anyone knows who has experimentally bred *Drosophila.* A half-pint bottle with 80 cc. of banana-agar will support many more than all the larvae one or two or three females will produce in 8 days. This is indicated by the absolute figures in the fifth column of Table 1. The bottles containing 9 mated pairs at the start produced 2117 offspring imagoes. Therefore the food was sufficient to allow at least that many larvae to come through successfully. But in the bottles with 1, 2, or 3 mated females the absolute number of larvae was much less than this. Now the drop from the one pair to the two pair bottles in rate of reproduction was 8.8. To prove that this cannot be explained by larval crowding and elimination we have only to multiply 21.4 the rate per ♀ day in the 1 pair bottles, by 89 the number of ♀ days in the 2-pair bottles. This will give the number of imagoes which would have been produced in the 2-pair bottles had the reproduction rate of the

1-pair bottles held there. The result is 1904.6. *But this is 200 or more less than the absolute number produced in various other bottles.* Therefore the drop in the rate from the 1-pair to the 2-pair bottles cannot be explained on the supposition of larval crowding and elimination. Possibly this factor comes into play in the higher densities, though there is no evidence as yet that such is the fact.

That density of population influences fecundity in the same sense that it is here shown to affect rate of reproduction was demonstrated with fowls more than a decade ago.[11] It was found that there was a steady decline in mean annual fecundity in Barred Rock pullets as one increased the number of birds per year from 50 to 100 and to 150. The experiments were so arranged that the square feet of floor space per bird (that is to say the density of population in Farr's sense) was substantially identical in all three sets—exactly so in the case of the 50 and 100 bird pens, and only slightly less so in the 150 bird pens. We cannot go into the details of these careful and critical experiments with poultry here, but would point out that they indicate that there is apparently another element involved in the case besides mere physical density, which element in our ignorance we may perhaps tentatively call psychological. Apparently there is a physiological effect of keeping large numbers of individuals together in a confined area or space, even though the amount of space or area *per individual* is identically the same in the large crowds of individuals as in the small crowds.

In general there can be no question that this whole matter of influence of density of population, in all senses, upon biological phenomena, deserves a great deal more investigation than it has had. The indications all are that it is the most important and significant element in the biological, as distinguished from the physical, environment of organisms.

Summary.—In this paper it is shown that the rate of reproduction of *Drosophila* during the first 16 days of its imaginal life varies inversely with density of population, in an extremely exact manner, in accordance with the following equation which is essentially an inverse form of Farr's Law,

$$\log y = a - bx - c \log x.$$

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ANIMAL EVOLUTION

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There are few lines of scientific speculation of such general interest as the problem of the evolution of the varied assemblage of types included in the animal kingom, while at the same time a logical and detached consideration of this problem is rendered exceedingly difficult both because of the isolated position occupied by many of these types and by the conviction we all have that man must represent the highest of them.

But while undoubtedly man is the most efficient and the dominant member of the animal kingdom it by no means necessarily follows that he and his fellow vertebrates are from the strictly biological standpoint the most perfect.

In the following pages I shall indicate, as briefly as possible, a line of reasoning whereby all the various animal types are brought into correlation with each other, and their evolution is shown to be not evolution in the sense of a progressive development from a lower type to a higher, but instead the gradual acquisition of increasing economic efficiency through the progressively greater and greater departure from biological perfection, correlated with the gradual loosening of the bonds by which the most perfect type is economically handicapped.

In a very large and important group of animals, the Protozoa, the body is composed of a single cell; but in all animals except the Protozoa the body is composed of a very large number of cells which are differentiated to serve definite purposes.

All multicellular animals begin life as a single cell. As at this stage they are in this respect comparable in structure to a protozoan, in later life they may be assumed to represent an advance over the protozoan type.

The original cell giving rise to a multicellular animal typically divides into two, four, and eight similar cells, the cleavage taking place in three planes each at right angles to the other two; each of these eight similar cells continues to divide until a hollow ball is formed of numerous cells all of which are alike. Such a structure is represented by a blastula, but no adult animal (? except *Salinella*) which can be regarded as the equivalent of such a stage is known.

The hollow sphere with its increasingly thin walls due to repeated cell divisions becomes mechanically unstable and collapses, forming a cup with a two layered wall, the inner lining being composed of the inturned cells of the collapsed portion.

There is now a differentiation of the cells into an external and an internal layer, but each of the cells in both layers is the equivalent of all the others in the same layer. A definite axis has appeared, passing through the center of the mouth of the cup (the gastrula opening) and through the opposite pole.

Since in every plane passing through the central axis the body wall is just the same as it is in every other plane the continued development of such a typical gastrula would result in the formation of a radially symmetrical animal developed about the axis of the gastrula as a center.

The logical conclusion is that those animals which develop radially about the original axis are biologically the most perfect of all animals since they continue without interruption into the adult stage the original developmental plan.

There are two groups of animals which develop in this way. In one of these, the sponges, the larva grows into a sort of radial mass colony or imperfectly integrated community of cells with little division of labor or unified life and no definite organs. In the other, the cœlenterates, the gastrula grows into an adult which resembles it in all its essential features, though innumerable minor refinements are added.

The sponges, being without any definite structures, indeed little more than proliferated masses of slightly differentiated cells, increase in size by simply growing out in all directions from the main axis and usually also upward.

The cœlenterates retain the definite body form of the original gastrula which with increase in size rapidly weakens mechanically in spite of the development of a series of internal buttresses; furthermore, they have definite organs, such as tentacles, which must be supported by the body wall, and they must be strong enough to capture and to hold the organisms which serve them as food. They are therefore limited to a definite and relatively small body size, excepting in a few cases of extreme specialization. But just as in the sponges there is no definite limit to the original growth impetus. In response to this the original polyp, having reached its full size, gives off a bud which grows into another polyp and this process may be continued indefinitely so that enormous arborescent or other colonies are formed.

Both the sponges and the cœlenterates continue in the same way to increase in bulk throughout life, the former by simple augmentation of the mass as a whole, the latter by the indefinite reduplication of the original unit. The colonial habit is a fundamental attribute of all cœlenterates, though many of them have secondarily become solitary through the disintegration of the colony coupled with an increase in size of the individuals.

Among the cœlenterates the solitary habit is commonly derived from the colonial through the process of strobilization. The original polyp elongates and from it plate-like units (ephyræ) one by one separate off by transverse fission distally which develop into large free living adults.

The original polyp which gives rise to the strobila is always at first bilaterally symmetrical, with two tentacles opposite each other. In the simpler cœlenterates the gastrula becomes the mouth of the adult, but in the more specialized types a considerable part of the external surface about this opening is drawn into it during growth forming a gullet tube.

Bearing these two facts in mind it is easy to reconstruct the path by which the so-called higher animals were evolved from the cœlenterate type.

Strobilization in the form characteristic of the cœlenterates became modified by the retention of the original bilateral symmetry and its accentuation, coupled with a more or less vague differentiation into a dorsal and a ventral surface, and the retardation of the liberation of the units formed, which are not detached until after sexual maturity has been reached, and furthermore have to a considerable degree lost their individuality, sharing a common nervous and excretory system. This condition is represented by the tape-worms, at the present time occurring only as parasites.

The next step was the further reduction in the individuality of the segments and their retention throughout life, resulting in the formation of a definite elongate jointed body composed of a series of similar segments in which a dorsal and a ventral surface are clearly distinguishable. Such a condition is represented by the annelids.

The unified body thus formed by the consolidation of what were originally separate and distinct entities in an animal colony which have now quite lost their individuality and become mere segments was still further perfected by the grouping of these segments into three units, the head, most unified, controlling and directing, the thorax, less completely unified, locomotor, and the abdomen, largest and least unified, containing the vital organs, as seen in the insects, or by a marked tendency to compress the body within the compass of a single group of firmly consolidated units, as seen in the crustaceans. In both cases the consolidation of the body was accompanied by a marked increase in the differentiation of the dorsal and ventral surfaces, and an extension of the former at the expense of the latter.

The body of the insects and crustaceans was thus derived from the colonial cœlenterates (as represented by the strobila) through the cestode and annelid types by the reduction of the colonial units to segments with progressively decreasing individuality accompanied by the modification of the radial symmetry first to a symmetry in which the body is the same on either side of two planes at right angles to each other (as in the polyp giving rise to the strobila), then to a similar but much more pronounced symmetry of the same kind with a faintly indicated dorsal and ventral surface (as in the cestodes), later to a bilateral symmetry with well differentiated dorsal and ventral surfaces, the former usually somewhat more extensive than the latter (as in the annelids), and finally to a very pronounced bilateral symmetry with the dorsal surface always greatly in excess of the ventral (as in the crustaceans and insects). The two tentacles of the polyp from which the strobila develops persisted as the lateral extensions of the proglottides in the tape-worms (which sometimes reach a considerable length), and these reappeared as the parapodia of the annelids, and in their most perfected form as the legs and other segmental appendages of the arthropods.

Strobilization comparable to the type characteristic of the cœlenterates also occurs in another form. The units, instead of budding off in a linear series, are formed on the walls of the interior of a closed polyp which finally disintegrates, liberating all the enclosed units as free living individuals. These have a symmetry which is apparently bilateral, but always shows marked radial features. Such a development is found in the liver fluke and its allies, though today in no non-parasitic forms.

This method of growth by which young are actually seen to be budded off from the interior of a closed sack developed from the ciliated larva, combined with the more or less extensive infolding within the gastrula mouth of part of the outer body wall as seen in the more specialized cœlenterates, furnishes a clue to the origin of the unsegmented animals.

In those colonial cœlenterates with division of labor the polyps are divided primarily into nutritive, reproductive, and "defensive," but the last are characterized by an abundance of cells containing an excretion and a coiled tubule, and some or other of them are constantly discharging the included liquid. They are probably fundamentally excretory and only secondarily adapted for defensive purposes.

It is easy to imagine that in the early stage of some cœlenterate type the infolding of the outer cell wall at the gastrula mouth was sufficient to bring within the inner cavity the region from which the new polyps are budded off, and to assume that there then were given off internally three buds, one of the nutritive (sack-like), one of the reproductive, and one of the excretory type, which developed into the perivisceral, the gonadial, and the nephridial cœlome.

The so-called cœlomate animals with unsegmented bodies, none of which, in sharp contrast to the segmented animals, have any true appendages, thus represent a colonial cœlenterate in which the colonial development has been transferred within the original unit; and the development of this internal colony has reached its maximum in the cephalochordates and the vertebrates.

The correctness of the interpretation of the segmented animals as primarily consolidated colonies is indicated by the retention in all the included groups—the cestodes, annelids, crustaceans, echinoderms and insects—of some form of asexual reproduction, budding, budding and fission, fragmentation, polyembryony or parthenogenesis which, though greatly reduced in the more specialized types, especially in the insects, may be traced back to the budding and budding and fission characteristic of the cœlenterate colonies.

The correctness of the interpretation of the unsegmented animals (except those with gill apertures) as ingrown colonies is indicated in all the included groups—the priapulids, sipunculids, molluscs, nemerteans, phoronids, brachiopods, and chætognaths—by the entire absence of asexual reproduction of any kind, this having been rendered impossible through the transference of the colonial development to the interior.

For the sake of clearness no mention has been made of the fact that the annelids, crustaceans, echinoderms and insects, belonging to the segmented series, also possess cœlomic structures and are to a certain extent therefore doubly colonial. In the annelids the two structural remnants of the primitive colony are about equally balanced. In the crustaceans and insects the cœlomic structures have almost disappeared in favor of the development of the segmentation. In the echinoderms the body is reduced to five half segments while the cœlome is highly developed.

With the segmentation and the development of the cœlome considered as derived from a fundamentally colonial habit there is no fundamental body form nor structure in the bilateral animals which cannot be traced back to an origin in the colonial cœlenterates. The cestodes and the trematodes represent the transition forms in which we see the colonial habit and the radial symmetry of the cœlenterate type breaking down and passing into the solitary habit and bilateral symmetry of the "higher" animals in which little or no traces either of the primitive radial symmetry or of asexual reproduction remain.

These numerous forms, the annelids, crustaceans, echinoderms and insects on the one hand and the priapulids, sipunculids, molluscs, nemerteans, phoronids, brachiopods, chætognaths, enteropneusts and vertebrates on the other are so inextricably entangled in a network of lines connecting each more or less definitely or vaguely with all the others and even crossing the gap between the two groups that it is quite beside the mark to speak of evolution in connection with them. Each phylum represents a sort of crystallization center about which a greater or lesser number of forms all showing but little deviation from a fixed type are closely grouped within a definite but broad and general structural complex.

Some of these crystallization centers or foci lie close together, as the crustaceans and the insects, while others are quite isolated, like the nemerteans, phoronids, brachiopods, chætognaths and echinoderms. They are not now connected by any intermediate forms, and there is no evidence that they ever were. Each represents a stable and economically efficient, circumscribed and distinctive, combination of structural characters crystallized out of a wide range of possibilities resulting from the disintegration of the cœlenterate type which has, so to speak, gone into solution.

Evolution, in the sense of progressive specialization, has been conclusively demonstrated within many groups, but always within restricted sections of these groups. There is no evidence that the larger groups, or the broader divisions within these groups, succeeded each other in any such way. The appearance of all the major multicellular animal types was probably simultaneous or nearly so, the original progenitor giving rise continually to innumerable variations in form, symmetry and organization most of which proved economically impossible, but many of which, proving suitable to meet a certain range of conditions, formed centers about which the various phyla were developed.

Whereas the animals between the cœlenterates and the enteropneusts are commonly considered as representing progressive evolution, this is not so in fact. It is true that they represent increasing bodily efficiency, but this bodily efficiency results from progressive structural deficiencies which throw into relief the features upon which this bodily efficiency is based. All the essential features of the bilateral animals preëxist in some form or other in the cœlenterates, but they can only become effective economically through the partial or more or less complete inhibition of others which here hold them in check.

Unimportant as they seem to most morphologists the possession of gill apertures characterizes a most interesting group of diverse animal types, the pterobranchiates, balanoglossids, tunicates, cephalochordates and vertebrates. In all of these except the balanoglossids there is a sudden return to the fundamental colonial habit in a modified form, as if the appearance of gill apertures marked the beginning of the senescence of the animal line as a whole which from that point on recapitulated, or rather paralleled, its earlier stages. In the cephalochordates and vertebrates the internal colony or cœlome becomes enormously subdivided, recalling more or less closely the conditions in the annelids; on the other hand in the tunicates there is practically no cœlome, but the formation of colonies by budding, which are often highly complicated and with division of labor, reappears. In the pterobranchiates (*Rhabdopleura* and *Cephalodiscus*) the formation of colonies by asexual budding is developed to a high degree.

From the preceding discussion it would seem that the only way by which the numerous and varied animal types can be made to fall into any evolutionary plan is to consider the radially symmetrical colonial cœlenterates as representing the highest degree of purely biological perfection, and all other animals (except the sponges which are their equivalent) as derived from them through the appearance of various defects which had the anomalous result of leading to increased bodily efficiency.

THE CORRELATION OF TIME UNITS AND CLIMATIC CHANGES IN PEAT DEPOSITS OF THE UNITED STATES AND EUROPE

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U. S. Dept. of Agriculture, Washington, D. C.

Read before the Academy April 25, 1922

This paper is a continuation of a previous publication[4] which contained a discussion of the evidence in peat deposits of the United States regarding climatic changes since the last Ice age.

For purposes of comparison a brief account in tabular form is presented to show certain correlations between peat deposits in this country with those of the continent of Europe.

1. A correlation of American with European *glacial drift sheets* has been made by Leverett.[8] By considering with Leverett the early and late Wisconsin Glaciation contemporaneous with the Würm glaciation it is possible to correlate the corresponding minor glacial stages.

The three readvances of the ice sheet that produced the morainic systems of the late Wisconsin: the Valparaiso-Kalamazoo, Lake Border, and Port Huron may be regarded as corresponding to Penck and Brückner's[10] cycle of mountain glaciations of the Alps: the Bühl, Gschnitz, and Daun-stadia—and to those of the Scandinavian continental ice-fields: DeGeer's[7] Dani-, Goti-, and Fini-glacial stadia.

2. *The Stratigraphic Study of Peat Deposits* in the United States has furnished valuable data as to the botanical composition of types of peat and their economic value. It has been shown also that the material formed under wet conditions differs in botanical composition and texture from that formed during dry periods, and that the study of separate and distinct layers in deposits formed within the main morainic systems during the glacial retreat serves to throw much light on the character of the past climatic changes.

Evidence is now presented by means of the position of conspicuous layers of forest peat, fibrous sedge and reed peat, and colloidal peat, to show the equivalence of peat deposits which were formed in the respective morainal belts of the United States and Europe. The greater structural complexity, it will be noted, is confined to areas within the oldest of the morainic systems, while within the later and more recent positions of the ice border the deposits are younger in age and consist of only a few layers of peat. The lack of structural diversity in the uppermost layers of the older deposits is related probably to the distance of those deposits from the direct climatic influence of the later glacial sub-stages.

TABLE I

CORRELATION OF GLACIAL, CLIMATIC, AND LIFE STAGES SINCE THE LAST ICE-AGE IN THE UNITED STATES

GLACIAL SUB-STAGES	GREAT LAKES STAGES	CLIMATIC CHANGES	ESTIMATED TIME	ANIMAL REMAINS	HUMAN REMAIN	VEGETATION SUCCESSION
8. Present	Lake Michigan 580 ft.		A.D.			Forest Spagnum bogs and marsh
7. Recent (post-glacial)	Post-Nippissing	Rising dry	—o— B.C.			
						Forest
6. Port Huron	Nippissing 530–590 Champlain Sea, Niagara Algogum 608–609	Temperate, humid	7000–5000 B.C.		Mound builders?	Marsh
morainic system	Lundy-Duluth 615 ft.	Temperate		Mastodon		
		Dry				Forest
5 Lake Border	Whittlesey-Chicago-Saginaw	Cool, humid			Flint implements	Marsh
morainic system	Maumee 805 ft.	Warm, dry	11000–8000 B.C.			
						Forest
4. Valparaiso	Late Wisconsin drift	Warm, humid	18000–16000 B.C.	Mastodon		Marsh
Kalamazoo morainic system	Loess formed	Cool, windy, dry				Steppe?
		Temporary amelioration				Low shrubs?
3. Bloomington		Cold, humid	25000 B.C.			
morainic system 2. Shelbyville moraines		Cold	28000 B.C.			Tundra?
1. Wisconsin driftborder		Arctic	35000–30000 B.C.			

To the practical worker it is obvious that a better understanding of the structural differences in peat deposits implies a more correct estimate of their economic value.

The profile sections visualize the stratigraphic relationship of American and European peatland and they suggest the value of a more detailed

TABLE II

STRATIGRAPHIC FEATURES OF PEAT DEPOSITS IN THE U. S.

	I CANTON, OHIO	II KENT, OHIO	III KANKAKEE, IND.	IV DISMAL SWAMP, VA.	V EVERGLADES, FLA.	VI MANTUA, OHIO	VII FREDMONT, IND.	VIII SANILAC, MICH.	IX ROME, N.Y.	X CHASSEL, MICH.
8.	Sedge	Forest	Forest	Forest	Sedge	Forest	Sedge	Shrub	Forest	Forest
7.	Sedge	+ Sphagnum	+ Macerated	+ Macerated	+ Macerated		+ Macerated	Sedge	+ Sedge	+ Sphagnum Sedge
—	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —
	?	Sedge	Sedge-Reed	Forest	Sedge	Forest	Colloidal	Forest	Forest	Forest
—	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —
6.	Sedge + Hypnum	Macerated	Sedge Macerated	+ Macerated	+ Macerated		+ Macerated	Sedge	Hypnum	
								Macerated		
—	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —
	Sedge	Forest	Shrub + Sedge	Forest	Sedge	Forest	Colloidal			
—	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —
5.	+ Hypnum	Sedge	Sedge	+ Macerated	+ Macerated	Sedge	+ Macerated Sand			
						Macerated	Macerated			
—	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —
	Reed	Forest	Forest	Forest	Sedge					
—	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —
4.	Macerated Clay	Sedge Macerated								
	Macerated									
—	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —
—	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —	— — —
3.										
2.										
1.										

study of peat utilization in Europe in order to determine the best practices necessary to a successful solution of the peatland problem in this country on related peat deposits.

3. The common *time* relationship is confirmed by a series of correlations. It is based on the similarity between the groups of morainal systems as the great, basic divisions of time, and on the loess deposits resting upon the moraines of the Würm-Bühl interstadium and the early Wisconsin drift, on the glacial lake stages of the Great Lakes and the Baltic, on the series of climatic changes, and on the remarkable association of the layers of peat material in the deposits which are referable to the same glacial substage; they mark the lesser divisions of time in the different geographic regions. These events were not local phenomena but well-marked stages

TABLE III

CORRELATION OF GLACIAL, CLIMATIC, AND LIFE STAGES SINCE THE LAST ICE-AGE IN EUROPE

GLACIAL SUB-STAGES	BALTIC LAKE STAGES	CLIMATIC CHANGES	ESTIMATED TIME	ANIMAL REMAINS	HUMAN* REMAINS	VEGETAION SUCCESSION
8. Present 7. Recent (Post-glacial)	Baltic o.m.	Rising, dry, cool, moist	A.D. —o— B.C. 1800 B.C.–300 A.D.		Iron age	Forest Younger sphagnum bogs
		Warm, dry	3000–1800 B.C.			Forest Heath (beech, fir)
6. Daun	Litorina sea 0–22 m.	Warm, moist	4000–3000 B.C. 7000–5000 B.C.		Shell heaps Neolithic	Older sphagnum
Fini-glacial		Continental			culture Lake dwellers	bogs
					Azilian	Forest (oak)
5. Gschnitz			10000–8000 B.C.		Paleolithic	
Goti-glacial	Ancylus Lake 66–175 m.				culture	Marsh
						Forest (birch, pine)
4. Bühl		Severe Continental	18000–16000 B.C.	Steppe horse	Magdalen	Marsh Steppe
Dani-glacial	Yoldia sea 300 m.	Loess formed		Reindeer	culture	Dryas Steppe
		Temp. amelioration, Warm, arid				Low shrubs (Aspen willow)
3.				Mammoth Reindeer	Solutrean	Sub-arctic
2.					culture Aurignacean	Tundra
1. Wurm drift border		Arctic	30000 B.C.		culture	

*Data taken in large part from Osborn (9)

Mousterian culture

which occurred nearly simultaneously over a large part of northern America and northern Europe. The combined evidence suggests that it is possible to date with fair exactness the general course of events since the last Ice-age. This chronology may furnish important data bearing on different branches of science.

(*a*) The estimate on the duration of Post-Wisconsin glacial time in the United States is based chiefly on the recession of the waterfalls of the Niagara. The figures for the larger subdivisions fall within the estimates made by Reeds,[12] Coleman,[3] Taylor,[8] Chamberlain and Salisbury[2].

TABLE IV

PROFILE SECTIONS FROM PEAT DEPOSIT OF EUROPE

	XI	XII	XIII	XIV	XV	XVI	XVII
	GERMANY	ENGLAND	AUSTRIA ALPS	SWEDEN	SWEDEN	RUSSIA	RUSSIA
8. 7.	Sphagnum	Top removed Sphagnum	Forest + Sphagnum	Sphagnum	(Forest) + Sphagnum	(Forest) + Sphagnum	(Forest) + Sphagnum
—	Heath	?	Shrub forest	Forest shrub	?	Forest	Forest
6.	Sphagnum	Sphagnum	Sphagnum	Sedge-Reed	Sphagnum	Sedge	Sphagnum
—	Sedge or						Hypmum
—	Forest shrub	Forest	Forest	Reed	Forest shrub	Forest	Reed
5.							Sedge
—	Sedge + Reed	Sedge + Reed	Reed	Macerated	Sedge Macerated	Macerated	
—	Reed or forest	Reed and drifted timber	Reed	Macerated			
4.	Macerated	Hypnum	Hypnum				
—	Macerated	Macerated					
—							
3.							
2.							
1.							

(*b*) The time units of measurement of Penck,[10] DeGeer,[7] Wahnschaffe[15] and Brooks[1] are more or less similar with the dates delimiting the American stages.

4. The correlation of past *climates* in the United States with those of the European last glacial period has not been worked out on a meteorological basis but there is sufficient information at hand of considerable interest for comparison with the conclusions based on European evidence.

The results of the correlation agree in showing that, although the evolution of climate has been continuous, there have been several climatic waves since the last glacial period, each of relatively long duration. The climatic pulsation, resembling a climatic belt, consisted of a cold and humid period bordering the glacial readvance, followed by a warmer, drier period in the accompanying interstadial time, with a tendency toward optimum conditions which favored the growth of forests. It is quite probable that the last climatic wave or belt, possibly in diminishing intensity, continues to shift northward at the present time.

5. In the past sixty centuries the power of nations has been moving rather steadily northward, and the higher production of agricultural crops has occurred in general near the northern limits of distribution. It is significant that this northward tendency in agricultural and social development seems to be continuing and that it appears to be correlated with practically the same course of events which is shown in Tables I to IV.

PROFILE SECTIONS OF PEAT DEPOSITS

I, II, VI and VII from *Botanical Gazette*,[5] **72**, 57–89, 1921. (4)

III. Profile of Kankakee Marsh near Crumstown, Indiana.

IV. Cross Section of Dismal Swamp, Virginia, generalized from soundings and exposures along the feeder ditch and the Dismal Swamp canal.

V. Cross section of Torry Island in Lake Okeechobee, Florida, also near Okeelanta, about five miles south from the lake shore.

VIII. Sounding on peat experiment sub-station near Sanilac, Michigan.

IX. Profile along barge canal near James street bridge at Rome, N. Y.

X. Sounding in central location of peat deposits at Chassel-Klingville, Michigan.

XI. After Weber. (16) XII. After Rankin. (11)

XIII. After Shreiber. (14) XIV. After Sernander. (13)

XV. After Erdtman. (6) XVI, XVII. After Docturowski. (5)

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THE SPECTRAL ENERGY DISTRIBUTION AND OPACITY OF WIRE EXPLOSION VAPORS

By J. A. Anderson

Mount Wilson Observatory, Carnegie Institution of Washington

Communicated, May 22, 1922

It has been shown[1] that a fine wire, through which is passed the discharge of a large high voltage condenser, vaporizes so rapidly that the phenomenon is best described as an explosion. The actual time required for the wire to pass from the solid to the vapor state is probably less than one millionth of a second. If the explosion is made to take place in the space between two parallel planes from 2 to 3 millimeters apart, such as a slot in a block of wood, its spectrum is continuous, the lines of the metal exploded appearing as absorption lines. The maximum intensity of the light is very great, being of the order of that to be expected from a black body at a temperature of about 20,000° K.

A preliminary study of the spectral energy distribution given by the explosion of iron wires has been made, using a quartz spectrograph with a vacuum thermo-couple[2] and galvanometer. The value of $E\lambda d\lambda$ increases very rapidly from the infra red to about $\lambda4300$; from this point to the ultra violet the great groups of iron absorption lines cause the values of $E\lambda d\lambda$ to oscillate considerably but its general tendency is upwards and the highest values are reached between $\lambda2600$ and $\lambda2150$. From $\lambda2150$ to $\lambda1990$ the values fall off rapidly, in part, perhaps, due to the absorption of the quartz lenses and prisms, and to that of the air.

Spectrograms of the explosions of the following metals have been made using a one meter focus grating spectrograph: copper, silver, gold, magnesium, zinc, cadmium, aluminum, tin, lead, tungsten, iron and nickel. All of these give continuous spectra of the same general intensity most of them appearing rather more intense than iron on account of the numerous absorption lines of the latter. Copper, silver and gold are anomalous in that the pure wires of these metals will not explode properly when placed in a slot in a block of wood, the main discharge always seeking a path around the outside of the wooden block. By very slightly amalgamating the surface of the wires they explode normally, giving fine continuous spectra crossed by their own absorption lines, the lines due to Hg not having been observed.

That a layer of these metallic vapors only a few centimeters thick should give a continuous spectrum is somewhat surprising, especially so, since the average pressure is certainly much lower than it was earlier[3] thought to be. Professor Henry Norris Russell suggested that it would be of some interest to find out whether the vapors are transparent to radiation or not. The following experiment was accordingly performed: A short spark gap connected in series with the wire to be exploded was (*a*) placed immediately in front of the wire, so that the light from the explosion had to pass through the spark on its way to the spectrograph; (*b*) placed immediately back of the wire, so the light from the spark had to pass through the explosion vapors before reaching the spectrograph. The spark and explosion would be exactly simultaneous, since they were connected in series. Iron wires were used, and the spark terminals were made of brass in order to make use of the bright emission lines of zinc throughout the spectrum and those of copper in the extreme ultra violet.

Using the arrangement (*a*) the spectrogram showed the bright zinc and copper lines very distinctly superposed on the regular iron absorption spectrum. It was also clear that the iron lines were less dark than usual, that is they were partially filled up by the continuous background in the light from the spark. With the arrangement (*b*) no trace of the zinc or copper lines could be seen, the iron spectrum being exactly the same whether the spark gap was used or not, thus showing that the light from the spark can not pass through the explosion vapors.

This experiment proves that 4 centimeters of iron vapor as here used is perfectly opaque, but it does not show how far the light from the spark was able to penetrate the vapor before it was absorbed, in other words, the absorption coefficient has been shown to be fairly large but has not been measured. Other observations make it probable that a layer about 2 mm. thick is very nearly opaque—but further experimental work is required, and will be undertaken as soon as possible.

[1] *Proc. Nat. Acad. Sci.*, **6** 1920, pp. 42–43; *Astroph. J.*, **51** 1920, pp. 37–48.

[2] The thermo-couple was kindly placed at my disposal by Mr. Edison Pettit.

[3] *Astroph. J.*, **51** 1920, pp. 44–46.

Acknowledgment. Since my Note on the Definition of a Linear Functional was published in the February number, I have learned that the principal theorem stated on page 27 has been proved by Fréchet by means of Fourier series, and that it is also a special case of a theorem by F. Riesz. See *Trans. Amer. Math. Soc.*, **8**, p. 439, and *Math. Ann.*, **69**, p. 475. CHARLES A. FISCHER

Erratum. Page 145, line 20. For "the majority of the stars of" read *about 20% of the stars of the.*

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PROCEEDINGS
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SPACES WITH CORRESPONDING PATHS

By L. P. Eisenhart

Department of Mathematics, Princeton University

Communicated June 24, 1922

1. In a former paper (these Proceedings, Feb., 1922) Professor Veblen and the writer considered the geometry of a general space from the point of view of the paths in such a space—the paths being a generalization on the geodesics in Riemannian space. It is the purpose of this note to determine the spaces whose paths may be brought into one-to-one correspondence with the paths of a given space, and to consider the degree of arbitrariness involved in our analytical definition of the paths.

2. The equations of the paths in a space S_n are taken in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha dx^\beta}{ds\ ds} = 0, \tag{2.1}$$

where x^i $(i = 1, \ldots, n)$ are the coordinates of a point of a path expressed as functions of a certain parameter s the same for all the paths, and $\Gamma^i_{\alpha\beta}$ are functions of the x's such that $\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$. When the space is Riemannian, with the quadratic form

$$ds^2 = g_{ij}dx^i\, dx^j, \tag{2.2}$$

equations (2.1) define the geodesics of the space, the functions $\Gamma^i_{\alpha\beta}$ being the Christoffel symbols of the second kind formed with respect to (2.2). Thus in choosing (2.1) to define the paths of a general space we have singled out a parameter s which is to be the same for all paths. If we change the independent variable in (2.1), it is found that the resulting equations have the same form only when this parameter is $as + b$, where a and b are arbitrary constants.

3. Suppose we consider a second space S_n' and write the equations of the paths in the form

$$\frac{d^2x^i}{ds'^2} + \Gamma'^i_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'} = 0. \tag{3.1}$$

We say that corresponding points in S_n and S'_n are those for which the coordinates x have the same values in the two spaces. If a path P in S_n is to correspond to a path P' in S'_n, there must be a relation of the form $s = f(s')$ connecting the parameters s and s' at corresponding points. In order that the functions x^i of s' shall satisfy (2.1) and (3.1) we must have

$$\frac{f''}{f'^2} = \frac{\left(\Gamma^i_{\alpha\beta} - \Gamma'^i_{\alpha\beta}\right)\dfrac{dx^\alpha}{ds}\dfrac{dx^\beta}{ds}}{\dfrac{dx^i}{ds}} = \frac{\left(\Gamma^j_{\alpha\beta} - \Gamma'^j_{\alpha\beta}\right)\dfrac{dx^\alpha}{ds}\dfrac{dx^\beta}{ds}}{\dfrac{dx^j}{ds}} \quad (i,j = 1, \ldots, n). \quad (3.2)$$

From these expressions it follows that we must have

$$\left[\left(\Gamma'^i_{\alpha\beta} - \Gamma^i_{\alpha\beta}\right)\frac{dx^j}{ds} - \left(\Gamma'^j_{\alpha\beta} - \Gamma^j_{\alpha\beta}\right)\frac{dx^i}{ds}\right]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \; (i, j = 1, \ldots, n). \quad (3.3)$$

If it is desired that all the paths of S_n can be brought into one-to-one correspondence with those of S'_n, these equations must be satisfied identically by the quantities $\frac{dx^i}{ds}$, otherwise we should have differential equations of the first order satisfied by all the solutions of (2.1). The conditions that these equations vanish identically are

$$\Gamma'^j_{ii} = \Gamma^j_{ii}, \; \Gamma'^i_{jk} = \Gamma^i_{jk} \; (i, j, k \mp), \quad (3.4)$$

and

$$\Gamma'^i_{ii} - 2\Gamma'^j_{ij} = \Gamma^i_{ii} - 2\Gamma^j_{ij}, \; \Gamma'^i_{ik} - \Gamma'^j_{jk} = \Gamma^i_{ik} - \Gamma^j_{jk} \; (i, j, k \mp) \quad (3.5)$$

Equations (3.5) are equivalent to

$$\Gamma'^i_{ii} = \Gamma^i_{ii} + 2\varphi_i, \; \Gamma'^i_{ij} = \Gamma^i_{ij} + \varphi_j. \quad (3.6)$$

where the functions φ_i $(i = 1, \ldots, n)$ of the x's are defined by (3.6). By making use of equations (4.2) of the former paper which give the relations between the Γ's in one set of coordinates x and those for another set, we prove that the functions φ_i in (3.6) are the components of a covariant vector.

If we substitute the expressions for $\Gamma'^\alpha_{\beta\gamma}$ from (3.4) and (3.6) in (3.2) we have

$$\frac{f''}{f'^2} = -2\varphi_\alpha \frac{dx^\alpha}{ds}. \quad (3.7)$$

If we substitute in the right-hand member the expressions for the x's as functions of s for a given path, we get the relation between s and s' in order that these functions shall satisfy (3.1) also. Consequently (3.4) and (3.6) are sufficient as well as necessary that the paths of S_n and S'_n be in one-to-one correspondence.

We are in a position now to determine under what conditions the paths of S_n are defined also by equations (3.1). From our definition of paths it follows that s' must be the same for all paths just as s is in (2.1). From (3.7) it is seen that for this to be the case we must have

$$\varphi_\alpha \frac{dx^\alpha}{ds} = \psi(s), \tag{3.8}$$

ψ being the same function for all the paths. Differentiating and making use of (2.1), we obtain

$$\varphi_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \psi'(s), \tag{3.9}$$

where $\varphi_{\alpha\beta}$ are the components of the covariant derivative of the vector φ_α as defined in the former paper, that is

$$\varphi_{\alpha\beta} = \frac{\partial \varphi_\alpha}{\partial x^\beta} - \varphi_i \Gamma^i_{\alpha\beta}. \tag{3.10}$$

Consider first the case when ψ in (3.8) is constant, that is when (2.1) admits a linear first integral. From (3.9) it follows that S_n must be such that there exists a vector φ_α so that $\varphi_{\alpha\beta} + \varphi_{\beta\alpha} = 0$. When this condition is satisfied by one, or more vectors, the equations of the paths can be written in two, or more ways, in the form (2.1). Spaces which admit a family of parallel covariant vectors, considered by the author in a former paper (these PROCEEDINGS, July, 1922) are of this kind.

If $\psi' = \text{const.}$ in (3.9), equations (2.1) would have to admit a quadratic first integral. Another case of this sort arises when $\psi' = a\psi^2 + b$, where a and b are constants. Although these special cases are interesting and lead to types of spaces for which the equations (2.1) are not unique, it is clear that this is not possible for a general S_n. Similar results hold when ψ' is not constant. For on the elimination of s from ψ and ψ' in (3.8) and (3.9) we should obtain a relation between the first derivatives of the x's, which could not hold for all the paths. Hence:

For a general space the equations of the paths can be written in the form (2.1) in only one way, the parameter s being such that it may be replaced by as + b, where a and b are arbitrary constants.

4. The components of the curvature tensor in S_n are defined by

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^\alpha_{jl}\Gamma^i_{\alpha k} - \Gamma^\alpha_{jk}\Gamma^i_{\alpha l} \tag{4.1}$$

If we indicate by primes the functions for S_n', we have from (3.4) and (3.6)

$$\begin{aligned} B'^i_{jkl} &= B^i_{jkl} \; (i,\ j,\ k,\ l \mp), \\ B'^i_{ijk} &= B^i_{ijk} + \varphi_{kj} - \varphi_{jk} \; (i,\ j,\ k \mp), \\ B'^i_{jki} &= B^i_{jki} + \varphi_{jk} - \varphi_j \varphi_k \; (i,\ j,\ k \mp), \\ B'^i_{jji} &= B^i_{jji} + \varphi_{jj} - \varphi_j^2 \; (i,\ j \mp), \\ B'^i_{iji} &= B^i_{iji} - \varphi_{ji} + 2\,\varphi_{ij} - \varphi_i \varphi_j \; (i,\ j \mp), \end{aligned} \tag{4.2}$$

where $\varphi_{\alpha\beta}$ is the covariant derivative of φ_α. In these formulas there are no summations.

5. By definition the contracted tensor is given by

$$R_{ij} = \sum_\alpha B^\alpha_{ij\alpha} \tag{5.1}$$

Hence from the above formulas we obtain

$$\begin{aligned} R'_{ij} &= R_{ij} + n\varphi_{ij} - \varphi_{ji} - (n-1)\,\varphi_i\,\varphi_j \\ R'_{ii} &= R_{ii} + (n-1)\,(\varphi_{ii} - \varphi_i^2). \end{aligned} \tag{5.2}$$

Since $B^i_{jkl} = -B^i_{jlk}$, a skew symmetric tensor of the second order is defined by

$$S_{ij} = \sum_\alpha B^\alpha_{\alpha ij}. \tag{5.3}$$

Now we have

$$S'_{ij} = S_{ij} + (n+1)\,(\varphi_{ji} - \varphi_{ij}). \tag{5.4}$$

From (5.1), (4.1) and (5.3) we have

$$R_{ij} - R_{ji} = S_{ji}$$

Hence $S_{ij} = 0$ is a necessary and sufficient condition that R_{ij} be symmetric. This result was established by Veblen in a paper presented to the Academy, April 25, 1922.

Veblen (these PROCEEDINGS, July, 1922) has shown that the identity of Bianchi, namely

$$B^i_{jklm} + B^i_{jlmk} + B^i_{jmkl} = 0 \tag{5.5}$$

holds in the geometry of paths; an equivalent result was announced by Schouten (Jahr. Deut. Math. Ver., vol. 30, 1921, Abt., p. 76). In consequence of (5.5) we have from (5.3)

$$S_{ijk} + S_{jki} + S_{kij} = 0, \tag{5.6}$$

or in other form

$$\frac{\partial S_{ij}}{\partial x^k} + \frac{\partial S_{jk}}{\partial x^i} + \frac{\partial S_{ki}}{\partial x^j} = 0. \tag{5.7}$$

We have shown (*Bull. Amer. Math. Soc.*, 1922) that (5.7) is a necessary and sufficient condition that S_{ij} is expressible in the form

$$S_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}, \tag{5.8}$$

where φ_i is a covariant vector, any one component being arbitrary and the others being determined by quadratures. As an immediate consequence we have the theorem:

If S_n is any geometry of paths, spaces S'_n can be found whose paths are in one-to-one correspondence with the paths of S_n and such that the tensor R'_{ij} is symmetric; the determination of S'_n involves an arbitrary function of the x's.

6. For a Riemann space the tensor R_{ij} is symmetric. Consequently

from (5.4) it follows that if S_n and S'_n are Riemann spaces, we must have $\varphi_{ji} = \varphi_{ij}$, that is φ_i must be a gradient.

Suppose now that S_n is any space. From the preceding theorem it follows that we can find a space S_n'' whose paths are in one-to-one correspondence with S_n and such that $S''_{ij} = 0$. If S_n' is a Riemann space in correspondence with S_n, and consequently with S_n'', it follows from (5.4) that the vector giving the relation between S_n'' and S_n' is the gradient of a function φ. The algebraic consistency of equations of the form (cf. (5.3) former paper)

$$g'_{i\alpha} B'^{\alpha}_{jkl} + g'_{\alpha j} B'^{\alpha}_{ikl} =$$

leads to more than $n\ (n + 1)/2$ equations. Hence when the expressions for B'^{α}_{ikl} as given by equations of the form (4.2) are substituted in these equations of condition, we have more than $n(n + 1)/2$ algebraic conditions upon the $n(n + 1)/2$ second derivatives of φ. Consequently the functions B''^{α}_{ijk} for S_n'' are subject to conditions and hence S_n cannot be arbitrary, if its paths are to be in one-to-one correspondence with the geodesics of a Riemann space. It follows also from these remarks that the final theorem of the former paper does not give the complete characterization of geometries whose paths can be identified with the geodesics of a Riemann space.

7. Suppose that S_n is a Riemann space. If S_n' also is to be Riemannian, it follows from (5.4) that φ_i must be the gradient of a function φ. If the components of fundamental quadratic tensors of S_n and S_n' are denoted by g_{ij} and g'_{ij} respectively, we must have

$$\frac{\partial g'_{ij}}{\partial x^k} - g'_{\alpha j} \Gamma'^{\alpha}_{ik} - g'_{i\alpha} \Gamma'^{\alpha}_{jk} = 0,$$

which in consequence of (3.4) and (3.6) becomes

$$\frac{\partial g'_{ij}}{\partial x^k} - g'_{\alpha j} \Gamma^{\alpha}_{ik} - g'_{i\alpha} \Gamma^{\alpha}_{jk} - 2g'_{ij} \frac{\partial \varphi}{\partial x^k} - g'_{kj} \frac{\partial \varphi}{\partial x^i} - g'_{ik} \frac{\partial \varphi}{\partial x^j} = 0. \quad (7.1)$$

If g and g' denote the determinants of g_{ij} and g'_{ij}, we have (Einstein, *Ann. Phys.*, **49**, 1916, p. 796).

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} = \tfrac{1}{2}\, g^{ij} \frac{\partial g_{ij}}{\partial x^k} = \Gamma^{\alpha}_{\alpha k}. \quad (7.2)$$

Multiplying (7.1) by g'^{ij}, summing for i and j, and making use of (7.2), we obtain an equation from which it follows that

$$\varphi = \frac{C}{2\ (n + 1)} \log \frac{g'}{g}$$

where C is a constant.

Again, if we put $A_{ij} = e^{-4\varphi} g_{ij}$ and indicate by A_{ijk} the covariant derivative of A_{ij} with respect to the fundamental form of S_n, we obtain

$$A_{ijk} + A_{jki} + A_{kij} = 0,$$

which is the condition that the equations of geodesics admit the quadratic integral $A_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \text{const.}$ These results are due to Levi-Civita (*Annali,* **24,** 1896, pp. 273, 276; also *Annalen,* **54,** 1901, p. 188), who has found all the Riemann spaces which admit of geodesic representation. When in particular it is assumed that S_n and S_n' possess corresponding orthogonal systems, and these are taken as parametric we have

$$\Gamma^i_{ii} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^i}, \Gamma^i_{ij} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x_j}, \Gamma^j_{ii} = -\frac{1}{2g_{jj}}\frac{\partial g_{ii}}{\partial x_j}, \Gamma^k_{ij} = 0,$$

and the equations (3.4) and (3.6) are readily solved; giving

$$g_{ii} = \prod_{j=1}^{n}{}' \left|\psi_j - \psi_i\right|, g'_{ii} = \frac{C}{\prod\limits_{j=1}^{n}(\psi_i + c)}\,\frac{g_{ii}}{\psi_i + c},$$

where $j \neq i$ in the first product, ψ_i is an arbitrary function of x^i alone, and c and C are arbitrary constants (Pirro, *Rend. Palermo,* **9,** 1895, p. 169; also, Levi-Civita, l.c., p. 287).

NUMBER OF SUBSTITUTIONS OMITTING AT LEAST ONE LETTER IN A TRANSITIVE GROUP

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In an article published in *Amer. J. Math.,* **26,** 1904, p. 1, H. L. Rietz proved the theorem that in "a primitive group G of degree n of composite order g there are more than $g/(x + 1)$ substitutions of degree less than n, where x is the number of systems of intransitivity of the sub-group which leaves a given letter fixed." If G is a transitive group which is not necessarily primitive this subgroup G_1 is of degree $n - \alpha$, where α is not necessarily unity, and may be supposed to have α transitive constituents of degree 1 in addition to its transitive constituents whose degrees exceed 1. If $x + 1$ represents the total number of these transitive constituents the theorem quoted above remains true for every possible transitive group except the regular groups, as may be inferred from an abstract which appeared in *Bull. Amer. Math. Soc.,* **8,** 1902, p. 17, where the exception noted above is, however, not stated.

The object of the present note is to furnish a very elementary proof of this general theorem, as follows: If each of the substitutions of G_1 involves the average number of letters, viz. $n - x - 1$, the number of substitutions of G which omit at least one letter is $ng_1 \div (x + 1) = g \div (x + 1)$ g_1 being the order of G_1. This case evidently presents itself only when G is regular. In all other cases this number of substitutions must be greater since each such case can be constructed by taking successively one letter away from some one substitution in the idealized G_1' in which all the substitutions including the identity are of the same degree, and adding it to another. By each such operation the number of substitutions which omit at least one letter is increased, since the number of such substitutions is thereby first diminished by $\frac{n}{k} - \frac{n}{k+1} = \frac{n}{k(k+1)}$, where $k \geqq x + 1$, and then increased by $\frac{n}{l-1} - \frac{n}{l} = \frac{n}{l(l-1)}$, where $l \geqq x + 1$. As each such operation increases the total number of substitutions which omit at least one letter this number must be increased by the totality of such operations which lead from the idealized G'_1 to the real G_1. It should be noted that the increase due to one such operation may be a fractional substitution.

In the special case when G is multiply transitive G_1 must be transitive and of degree $n - 1$. In this case $g/(x + 1) = g/_2$ and by reducing one substitution of G'_1 to the identity we increase the number of substitutions whose degree is less than n by

$$\frac{n(n-2)}{2} - \frac{n}{2} + 1.$$

If G_1 contains substitutions whose degree is less than $n - 2$ the number of substitutions of degree $n - 1$ must be still larger. Hence it results directly that in a multiply transitive substitution group of degree n the number of substitutions of degrees $n - 1$ and $n - 2$ cannot be less than

$$\frac{g}{2} + \frac{n(n-3)}{2},$$

and this is the exact number of such substitutions when G involves no substitution besides the identity whose degree is less than $n - 2$ and only then.

The main theorem of this note may be stated as follows: *In every non-regular transitive substitution group of degree n the number of the substitutions which omit at least one letter exceeds g/x where x is the number of the transitive constituents in a subgroup composed of all the substitutions which omit one letter regarded as a group of degree n.* One reason for emphasizing this matter here is that the theorem is stated incorrectly even for the special case when the group is primitive on page 357 of Cajori's *History of*

Mathematics, 1919, as a consequence of a misprint in my "second report on recent progress in the theory of groups of finite order," *Bull. Amer. Math. Soc.*, **9**, 1903, p. 118.

PHOTIC ORIENTATION IN INSECTS

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The idea that orientation in organisms is dependent upon the relation in the physical and chemical states produced by the stimulating agent in symmetrically located receptors and the relation in the activity of the locomotor apparatus on opposite sides is widely accepted. This idea extends back to the botanist, Ray (1693), it was applied to lower animals by Verworn in 1894 and to higher animals including insects by Bohn in 1904 and Loeb in 1906. It has been referred to as the difference of intensity theory, the continuous action theory, the tropism theory, the Verworn theory, the Ray-Verworn or DeCandolle theory, the muscle-tension theory and Loeb's muscle-tension theory. We shall call it the Ray-Verworn theory.

Referring to insects, Bohn, Loeb, Garrey and others maintain that light affects the tonus of the muscles in proportion to the intensity of the illumination of the receptors with which they are connected and in such a way that when the eyes on opposite sides are unequally illuminated, the legs on one side move faster than those on the other resulting in turning until the eyes are equally illuminated, when the tonus of the legs on opposite sides becomes equal and the turning ceases. These authors maintain that whenever insects are oriented, the receptors on opposite sides receive the same amount of stimulating energy. They hold that orientation is the result of balanced action in the two halves of the body.

There are numerous reactions involved in the process of orientation of insects and other animals which are not in accord with these views. Some of these follow.

If one eye of the drone-fly, Eristalis tenax, is covered it deflects strongly toward the functional eye making circus movements, but after a certain time it orients fairly accurately, especially in a well defined beam of light. Similar results have been obtained in observations on various other insects by Radl, (1903), Holmes (1905), Carpenter, (1908), Dolley (1916), and Minnich (1919). The eyes on opposite sides are not equally stimulated when these insects with but one functional eye are oriented as is demanded by the Ray-Verworn theory. And the same may be said in reference to the fiddler-crab and a number of other forms that move sidewise toward the light.

In the robber-fly, Erax rufibarbis, with one eye covered the body assumes a tilted posture. The legs on the blind side are much more extended than those on the normal side and the feet consequently tend to move faster on the former than on the latter side, resulting in turning toward the normal side. Similar reactions are obtained if the lower half of one eye, the right e.g., and the upper half of the other eye are covered, i.e. they lean toward the right side and turn to the right; but if such specimens are laterally illuminated from the left, they will, under certain circumstances, turn to the left. Thus they turn toward the side on which the extension of the legs is greater, and in the direction precisely opposite to that which would obtain if it were in accord with the tonus hypothesis.

If Eristalis on the wing is illuminated from above or below it turns directly upward or downward. This turning can not be due to unequal illumination of the two eyes in accord with the Ray-Verworn theory for the two eyes may be continuously equally illuminated. Similar responses have been observed in Caprella (Mast, 1911, p. 224).

In light from two sources both Eristalis and Erax, go toward a point between the sources. The location of this point depends upon the relation of the luminous intensity of the light received from the two sources. The greater the difference the nearer the more intense illumination the point is located. When the insects are oriented in such a field of light the two eyes are not equally illuminated except when the light received from the two sources is equal. Consequently, when insects are oriented under natural conditions where they are nearly always subjected to light from various sources of unequal intensity, the eyes are rarely if ever equally illuminated. Under such conditions orientation is therefore not in accord with the Ray-Verworn theory as applied to insects by Bohn, Loeb, and others.

In specimens of Eristalis or Erax with one of the front legs removed orientation is practically normal. If such specimens are laterally illuminated they turn toward the light either to the right or to the left. If the front and the middle legs on the same side are removed, Eristalis deflects strongly toward the normal side, but after a few days it orients fairly accurately. Erax usually goes fairly directly toward the light at once, but only for a short distance after which it deflects rather sharply toward the normal side; then it ordinarily attempts to turn in the opposite direction, but it usually fails, and topples over. The movements of the legs are not well coördinated and there is consequently much difficulty in locomotion. Orientation in these specimens obviously can not be due to equal action in locomotor appendages on opposite sides as the theory in question demands. Specimens of Eristalis with the front and middle legs on one side removed and either eye covered may under certain conditions proceed fairly directly toward the light. If the direction of the rays is changed while they are thus proceeding toward the light, they reorient by turning either to the right or to the left. This shows that the movements of the legs on either side may be controlled by impulses originating in either eye. These responses are obviously not in accord with any theory of orientation that demands balanced or equal action either in the receptors or in the locomotor appendages on opposite sides.

In certain fire-flies the male, in response to a flash of light produced by the female, turns thru the required angle, no matter in which direction he may be going in reference to the location of the flash, and proceeds directly toward her. Thus after momentary illumination, he may turn in the total absence of light in any direction, thru angles varying from 0 to 180 degrees; and this turning is, under many conditions, a response to the illumination of but one eye. It is consequently evident that it cannot be accounted for by any theory which demands balanced or equal illumination of receptors on opposite sides as the basis of orientation.

The facts presented above show that there are a number of phenomena of fundamental importance in the process of orientation in insects that are not in accord with the Ray-Verworn theory. How can these phenomena be explained?

Insects have image-forming eyes, so that when an eye is exposed to a source of light an image is produced on the retina, and if the source is concentrated the image is very small, resulting in definite localization of the stimulus. Thus while the whole surface of the eye is illuminated the stimulus may be confined to a minute portion of the retina, the location of which depends upon the location of the source of light in relation to the longitudinal axis of the insect. If the light is in front of the insect the image is near the anterior edge of the retina; if it is to the side the image is on the lateral surface of the retina, etc. In general it is always directly below the ommatidium whose longitudinal axis is parallel with the rays of light.

The flash produced by a female fire-fly forms on the retina of the male a minute luminous image so that the stimulation is practically confined to a point. This point varies in location from the posterior to the anterior edge of the retina depending upon the axial position of the male in reference to the female, and it also varies in a dorso-ventral direction. There is then for every axial position of the male a definite location of the stimulus in the eye. The fact that in response to a flash of light produced by the female, the male turns after the flash has disappeared until he faces the female regardless of the extent of turning required, indicates that momentary stimulation of a point on the retina sets up a series of turning reflexes, and that the series set up varies in character and in extent with every point; so that if the point stimulated is located at the posterior edge of the retina the series of reflexes carries him through a large angle, the degree of turning in every case depending upon the location of the stimulus. Thus in the fire-fly orientation is the result of series of reflexes, the nature and extent of which is specifically related to the location of the stimulus. There is much evidence indicating that similar factors are involved in the process of orientation in other insects.

If in Eristalis or Erax with one eye covered the stimulus is localized at the posterior edge of the retina, the feet on one side move forward while those on the other move backward, the two front feet deflecting toward the side stimulated, the two hind feet from this side. If it is localized in the lateral portion both front feet move laterally toward the light, as do also the middle feet but to a less extent. If it is localized in the central part of the anterior surface of the eye the feet on both sides move forward and the insect does not turn. If it is localized at the antero-median edge it turns toward the covered eye.

This indicates that stimulation of a given region of the retina in one eye induces certain definite reactions in all of the legs on both sides, of such a nature as to tend to turn the insect toward the point stimulated just as it does in the fire-fly. But if there is only a momentary illumination Eristalis and Erax turn thru only a very small angle regardless of the location of the stimulus, whereas the fire-fly turns until it faces the source of light. However, if the illumination continues, then, as Eristalis or Erax turns, the image of the source of light on the retina travels forward and a continuous series of points extending forward in the eye become successively stimulated and the stimulation of each successive point induces reactions of such a nature as to produce turning toward the region stimulated and the result is continuous turning until the image reaches the anterior portion of the eye and the insect faces the light. Orientation is brought about by a series of reflexes (differential responses to localized stimulation) similar to the scratch reflexes in higher forms induced by stimulation of various points on the surface of the body. In photic orientation the nature of each series of reflexes depends upon the localization of the stimulus in the eye just as the nature of the scratch reflexes depends upon the localization of the stimulus on the surface of the body. This is what occurs if the stimulation is confined to one eye. If both eyes are simultaneously stimulated other processes are involved as is indicated in the following paragraphs.

If Eristalis is exposed to light from two sources at the same distance and equal in size and intensity and so arranged that the rays cross approximately at right angles at the place of exposure, it will turn until it faces a point half way between the two sources and then proceed toward this point. If the two sources of light differ in intensity it will turn until it faces a point nearer the more intense source and the greater the difference the nearer this source the point will be. Under the first conditions the two eyes are equally illuminated when the insect is oriented; under the second they are not; the region stimulated in one eye is more intensely illuminated and it is farther forward than that in the other eye, and the greater the difference in the intensity of the two sources, the greater the difference in the location and in the intensity of the illumination of the regions stimulated in the two eyes. Consequently the turning effect of stimulation of a given region of the retina in one eye is obliterated by simultaneous stimulation of the same region in the opposite eye, provided the stimuli are of the same magnitude, and by simultaneous stimulation of any other region in the retina of the opposite eye provided the stimuli in the two eyes bear the proper relation in magnitude. If the stimulus in one eye is located relatively farther forward than that in the other eye, the former in order to produce complete inhibition, must be stronger than the latter, if farther backward it must be weaker.

The elimination of the effect of stimulation in one eye by simultaneous stimulation in the other eye is not due to antagonistic action of the legs on opposite sides as demanded by the Ray theory of orientation as applied to animals by Verworn, Loeb, Bohn, and others. The elimination is due to the total absence of any appreciable effect of the stimulating agent on the muscles of the legs. When an insect is oriented in light, the light has no immediate observable effect on the muscles.

Photic orientation in these organisms is the result of series of coördinated reflexes in the legs of both sides specifically related to the localization of the stimulus in either eye and inhibition or modification of the effect of the illumination in one eye by simultaneous illumination in the other.

This view of the process of orientation is in full accord with all of the phenomena presented above. It accounts for orientation in insects with one eye covered or with legs on one side removed as well as for orientation in normal specimens. It accounts for orientation on the wing in a vertical plane. It accounts for orientation in Caprella and in the fiddler crab. It accounts, moreover, for the facts observed by Dolley (1916) in Vannessa with one eye covered that the degree of deflection toward the functional eye in a horizontal beam of light is independent of the intensity, and decreases as the edge of the beam is approached where it becomes zero and that the degree of deflection may, in illumination from a concentrated source directly above, be greater in weak than in strong light, if reflection from the back ground is eliminated.

Photic orientation in insects is as a whole adaptive. It is rarely injurious except under unnatural conditions, as, e.g., in light produced by a candle in a dark room. The reflexes resulting in orientation are similar to the scratch reflexes in the dog. How it is that they have come to be specifically correlated in character and extent with the localization of the stimulus in the eye is not known. They are instinctive in nature since they are largely independent of experience in the individual in which they occur and their origin is doubtless the same as that of other instinctive reactions.

A full description of photic orientation in Eristalis and Erax will be published shortly. This paper will contain an extensive bibliography including all of the literature referred to above.

HESPEROPITHECUS, THE FIRST ANTHROPOID PRIMATE FOUND IN AMERICA

By Henry Fairfield Osborn

American Museum of National History, New York

Read before the Academy, April 25, 1922

This communication to the National Academy, Tuesday April 25, 1922, was simultaneous with its publication in the American Museum *novitates.** A single small water-worn tooth, 10.5 mm. by 11 mm. in crown diameter, signalizes the arrival of a member of the family of anthropoid Primates in North America in Middle Pliocene time. The discovery is due to Harold J. Cook, consulting geologist, Agate, Nebraska.

The anthropoid Primate characters of the tooth are confirmed by another water-worn third upper molar previously found by William D. Matthew in the same beds but not described because it was not sufficiently distinctive. These two teeth establish the existence in the Pliocene period of a new and independent type of anthropoid, intermediate in the structure of its grinding teeth between the anthropoid ape and the human type. The animal is certainly a new genus of anthropoid ape, probably an animal which wandered over from Europe and Asia with the large south Asiatic element that has recently been discovered in our Pliocene fauna by Merriam, Gidley, and others. The generic name *Hesperopithecus* signifies *Pithecus* of the Western Hemisphere; its specific name *haroldcookii* is assigned in honor of the discoverer. The tooth actually resembles the human type more closely than it does any known anthropoid ape type; consequently it would be misleading to speak of this *Hesperopithecus* at present as like the known anthropoid apes; it is a new and independent type of Primate and we must seek more material before we can determine its relationships. It has been suggested humorously that the animal should be named *Bryopithecus* after the most distinguished Primate which the State of Nebraska has thus far produced. It is certainly singular that this discovery is announced within six weeks of the day (March 5, 1922) that the author advised William Jennings Bryan to consult a certain passage in the Book of Job, "Speak to the earth and it shall teach thee," and it is a remarkable coincidence that the first earth to speak on this subject is the sandy earth of the Middle Pliocene Snake Creek deposits of western Nebraska.

The geologic age of these two Primate teeth is now believed to be the same as that of Thousand Creek, Nevada, and Rattlesnake, Oregon, among the fauna of which *Pliohippus* is very abundant and varied; it also contains *Ilingoceras* and other strepsicerine antelopes of Asiatic affinity; it is the last American fauna in which occurred the rhinoceros, preceding the Blanco fauna in which the Asiatic brevirostrine *M. mirificus* first occurs.

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* Osborn, Henry Fairfield, "*Hesperopithecus*, the First Anthropoid Primate Found in America," *American Museum Novitates*, No. 37, April 25, 1922.

RECENT DISCOVERIES ON THE ANTIQUITY OF MAN

BY HENRY FAIRFIELD OSBORN AND CHESTER A. REEDS

AMERICAN MUSEUM OF NATURAL HISTORY, NEW YORK

Read before the Academy, April 25, 1922

This paper is an abstract of two lines of research recently undertaken by the authors which will be published under the titles: *Old and New Standards of Pleistocene Division in Relation to the Prehistory of Man in Europe,* Pliocene (Tertiary) and Early Pleistocene (Quaternary) Mammalia of East Anglia, Great Britain, in Relation to the Appearance of Man.***

At the April, 1921, meeting of the National Academy, Dr. Osborn ventured the prediction that a large-brained type of man would be found in the Pliocene. He was not aware that such a discovery had actually been made in the Upper Pliocene Red Crag deposits near Ipswich, England, by J. Reid Moir; in fact, Mr. Moir's discovery of Red Crag and of sub-Red Crag man was not accepted in England, and it was not until an unmistakable human industrial level was found at Foxhall, near Ipswich, in the summer of 1921, that this locality was visited by the French archaeologist Breuil, who announced this important discovery at the Archaeological Congress at Liége, in August, 1921. Dr. Osborn immediately planned to visit this locality and to make a careful survey and review of the animal life which surrounded Foxhall man. This review, fully set forth in the second paper above named, shows that Foxhall man—capable of making ten or twelve different kinds of flint implements, of providing himself with clothing, and of building a fire—sets an unmistakable Upper Pliocene date for the antiquity of man, in which he was surrounded by relatively primitive mastodons, rhinoceroses, saber-tooth tigers, and two species of elephants, a fauna closely similar to the Upper Pliocene fauna of the valley of the Arno River, near Florence, Italy.

More recent than the Foxhall industry is that of Cromer on the coast of Norfolk, discovered during the summer of 1921. Cromer is also treated as of Upper Pliocene age by British archaeologists, but it is unmistakably *Lower Pleistocene;* it belongs to First Interglacial time. To establish this second point, Dr. Osborn enlisted the coöperation of Dr. Chester A. Reeds, beginning in the month of September, 1921, and undertook an exhaustive examination of the old and new standards of Pleistocene division in Europe, namely, of Geikie, Penck, Brückner, and Leverett, ending with the recent work of Depéret and of De Geer. While these authorities do not agree as to causes or as to the duration of the Ice Age, a most important result of concurrent observation in England, France, Germany, Switzerland, and North America is that there were certainly within the Ice Age four, and possibly five, distinct periods of glaciation, with at least four interglacial periods, all embraced within the Quaternary. British geologists and at least one French geologist, Marcellin Boule, include the First Glaciation within Tertiary time, but all the other authorities named above regard the First Glaciation as the opening of Quaternary time. The latter view is the one adopted by Osborn and Reeds and is clearly set forth in a synthetic diagram which summarizes our present knowledge of the geologic succession, of the industrial phases, and of the geologic appearance of human types. This Table will be submitted to the coming International Congress of Geology at Brussels.

As a result of the researches summarized by the authors, we are now able to fix with considerable certainty the geologic level of the entire succession of human industries, namely, the Foxhallian, Cromerian, Chellean, Acheulean, Mousterian, Aurignacian, Solutrean, Magdalenian, and Campignian. We are also able to fix with considerable exactitude the geologic age of the successive races of men, i.e., Trinil, Piltdown, Heidelberg, Neanderthal and Cro-Magnon, which appear between Foxhallian and Magdalenian industrial times.

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[*] This paper, presented in abstract by Dr. Reeds before the Geological Society of America, is now in press in the *Proceedings* of the Society.

[**] This paper by Dr. Osborn will appear in full in the *Geological Magazine,* London.

ON THE MEASUREMENT OF A PHYSICAL QUANTITY WHOSE MAGNITUDE IS INFLUENCED BY PRIMARY CAUSES BEYOND THE CONTROL OF THE OBSERVER AND ON THE METHOD OF DETERMINING THE RELATION BETWEEN TWO SUCH QUANTITIES

By W. A. Shewhart

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Read before the Academy, April 25, 1922

In order to relate the physical and chemical properties of carbon to the microphonic properties of this material, it has been found necessary to make measurements on physical quantities which are influenced by primary causes beyond the control of the observer. In many problems of the physical and engineering sciences it is possible for the observer to control within narrow limits, the causes of variation of a quantity while it is being subjected to measurement. Certain problems arise, however, in these sciences as in the field of economics and biology wherein it is impossible to do this. In general let X_1 and X_2 represent two quantities related to others $U_1, U_2, \dots U_a;$ $V_1, V_2, \dots V_b;$ $W_1, W_2 \dots W_b$ in the following way:

$$X_1 = F_1 (U_1, U_2, \dots U_a, V_1, V_2, \dots V_b) \qquad (1)$$

$$X_2 = F_2 (U_1, U_2, \dots U_a, W_1, W_2, \dots W_b) \qquad (2)$$

where F_1 and F_2 represent unknown functional relations. In most physical experiments it is possible to hold the U's, V's, and W's, practically constant while a measurement is being made on either X_1 or X_2. In the last analysis, however, the U's, V's, can never be held constant and in many cases, particularly where these symbols represent molecular phenomena, the variations about their mean values may become quite large.

Thus, if X_1 and X_2 represent two microphonic characteristics of granular carbon which are functions of the physical and chemical properties represented by the U's, V's, and W's, it has been found necessary to apply certain statistical criteria to determine the nature of the cause complex controlling a single quantity such as either X_1 or X_2 and also to determine quantitatively the degree of relation existing between the two microphonic properties. As a result of such a study involving an analysis of thousands of measurements of the above type, certain conclusions which are of general interest have been reached in respect to the practical application of statistical methods in connection with physical measurements of this character.

In order to study the nature of the complex of causes controlling a single quantity such as X_1, one of the first problems is to determine whether or not the causes represented by the U's, and V's satisfy the following conditions: (1) That all of the causes, n in number, are effective at the time of each observation, (2) that the probability, p, that a cause will produce a positive effect is the same for all of the causes. (3) That the probability, p, remains the same for all of the observations, (4) that the effect, Δx, of a single cause is the same for all of the causes.

If these conditions are fulfilled, the distribution in X_1 can be represented by the successive terms of the expansion $(p + q)^n$ where the ordinates are separated at intervals of $2\Delta x$. For most of the problems it has been found convenient to compare the observed distribution with the theoretical distribution consistent with the above random conditions. In general, the following procedure has been followed: For each observed distribution two factors $k = \frac{\mu_3}{\mu_2^{3/2}}$ and $\beta_2 = \frac{\mu_4}{\mu_2^2}$ have been calculated and compared with similar factors consistent with the above mentioned binomial expansion, where the first four corrected moments of the observed distribution about the mean are represented by the symbols μ_1, μ_2, μ_3, and μ_4. It will be noticed that k and β_2 are independent of the units used in making the measurements.

For a symmetrical distribution k is always 0 and β_2 may vary between 1 corresponding to $n = 1$ and 3 corresponding to $n = \infty$. Irrespective of the values of p and q, it should be noted that β_2 does not increase beyond 4 for 10,000 causes and even for 100 causes the value of 4 is not exceeded except for conditions in which the skewness k is very large. In general large values of β_2 indicate either high skewness and few causes or that the complex of causes does not follow the random conditions consistent with the expansion of $(p + q)^n$.

As a result of variations due to sampling the standard deviations, σ_k and σ_{β_2} of k and β_2 are given in terms of the number of observations, s, by the following expressions $\sigma_k = \sqrt{\frac{6}{s}}$ and $\sigma_{\beta_2} = \sqrt{\frac{24}{s}}$ it being assumed that the distribution is practically normal. If, in practice, the values of k and β_2 are found to differ from 0 and 3 respectively by more than three times the standard deviations of k and β_2, it is practically certain that the distribution is either unsymmetrical or that the causes do not act at random as defined above. The method of application of these criteria depends in general upon whether or not the observed distribution is one involving attributes or variables.

In general for the case of attributes the procedure is as follows: The factors k and β_2 are calculated to determine whether or not they are significantly different from 0 and 3. If k and β_2 are different from 0 and 3, the values of p, q, and n may be calculated since $pn = \bar{x}$ and $pqn = \mu_2$ where $\bar{x}$ represents the average of the distribution, and a study of these values will indicate the probable reason for such variations in k and β_2 provided the distribution follows the random laws. If the observed distribution is consistent with the calculated values of p, q, and n, the "probability of fit"[1] between the observed distribution and the theoretical distribution representing the expansion of $(p + q)^n$ should be high.

In the case that the U's, V's and W's are variables, we start with unknown values of p, q and n, and in addition have the unknown value of ΔX which represents the effect of a single cause. It is obviously impossible to determine the values of p, q and n as in the case of attributes for among other reasons we cannot determine the origin of the distribution. In general, however, we may make use of the factors, k and β_2, since these are independent of the unit in which the measurements are made. Furthermore, we may make use of the criterion, establishing the "goodness of fit" between the observed distribution and a theoretical one corresponding to the various degrees of approximation[2] to the normal law.

To return to equations (1) and (2) it is evident that, if none of the causes are common, the correlation between simultaneous measurements of X_1 and X_2 will be practically zero. Furthermore, as the number of causes that are common increases we should expect an increase in the correlation coefficient under certain limiting conditions. Though it may not be possible to determine the number of causes that influence a variable quantity, such as either X_1 or X_2, it has been found possible, however, to determine the approximate value for the ratio, of the number of common causes to the total number of causes influencing either variable. To show this for the general case assumed in equations (1) and (2) let us start with the definition of the correlation coefficient r,

$$r = \frac{\Sigma x_1 x_2}{N\sigma_{x_1}\sigma_{x_2}} \tag{3}$$

where x_1 and x_2 represent the deviations from the mean values of X_1 and X_2 and σ_{x_1} and σ_{x_2} represent the standard deviations in these quantities and N represents the total number of observations made. Let us assume independence between the primary causes. If we expand equations (1) and (2) by Taylor's theorem and neglect thes econd and higher powers of the variations of the primary causes we can obtain values of the deviations x_1 and x_2 and of the standard deviations σ_{x_1} and σ_{x_2}, which, when substituted in equation (3) reduce to the simple expression

$$r = \frac{a}{a + b} \tag{4}$$

providing we add the further conditions that the standard deviations in all of the primary causes are equal and that the effects of all of the primary causes are equal. These conditions appear at first very limited, and in a practical case it is necessary at least qualitatively to check the assumptions underlying equation (4), but in general it has been found that a significant deviation in r may be taken to indicate a variation in the effects of one or more groups of primary causes, upon one of the variables and not upon the other.

A more complete account of this investigation giving experimental data, will be published elsewhere.

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[1] Pearson, K. "Tables for Statisticians and Biometricians" pp. (26–28) and *Phil. Mag.*, Ser. 5, **1**, 1900.

[2] Edgeworth, F. Y., *Camb. Phil. Trans.*, **20**, 1904 (36–65) and (113–141).

ELECTRONIC BOMBARDMENT OF NICKEL

By H. E. Farnsworth

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Communicated June 13, 1922

Studies of the secondary (emergent) electrons produced by electronic bombardment of metal surfaces have previously been made by various experimenters.[1] However, the fact that most of the results were obtained previous to the development of modern high vacuum technique combined with the failure of the small amount of recent work to agree with these older results, leaves the important questions of this problem still unanswered. It is well known that the number of secondary electrons depends upon the velocity of the primary (incident) beam, that more electrons leave the surface than strike it if the primary velocity is great enough, but the following characteristics of the various metals are still not definitely determined: (1) the magnitude of the secondary electron current as a function of the primary velocity; (2) the velocity distribution of the secondary electrons for any given primary velocity.

The present investigation is an attempt to obtain these characteristics for nickel. The type of apparatus employed was such that a primary electron stream, of uniform velocity and cross-section, could be made either to impinge on a nickel target or to pass into an absorbing Faraday cylinder. This was accomplished by moving the target, by means of a magnetic control, over a hole in the Faraday cylinder into the path of the primary stream. The secondary electrons which left the target were received by a conducting cylinder. This arrangement made it possible to move the target out to the side of the tube where it could be heated to bright red heat by high-frequency induction currents.

The ratio of the secondary to the primary electron current was studied, especially as affected by: (1) prolonged heating of the bombarded surface at bright red heat; (2) condition of the surface; (3) nature of the gas to which it had been exposed. This ratio, for low primary velocities (less than 260 volts), decreased with prolonged heating, and the general shape of the secondary electron curve was changed. A limiting form of this curve was obtained, see figure 1, which was independent of whether the apparatus had previously been exposed to air or hydrogen; so it seems safe to conclude that the final curve is one which represents the true characteristics of nickel. The number of secondary electrons was found to be independent of the roughness of the surface, but covering the surface with fine nickel gauze considerably reduced this secondary electron current. Secondary electrons were found when the primary velocity was as low as 0.2 volt. In the limiting curve, see figure 1, the ratio of the secondary to the primary current increased rapidly with increase of the primary velocity to a value of 0.15 at about 4 volts, then remained constant to about 9 volts, where it again increased, reaching a value of unity for a velocity of impact corresponding to 260 volts.

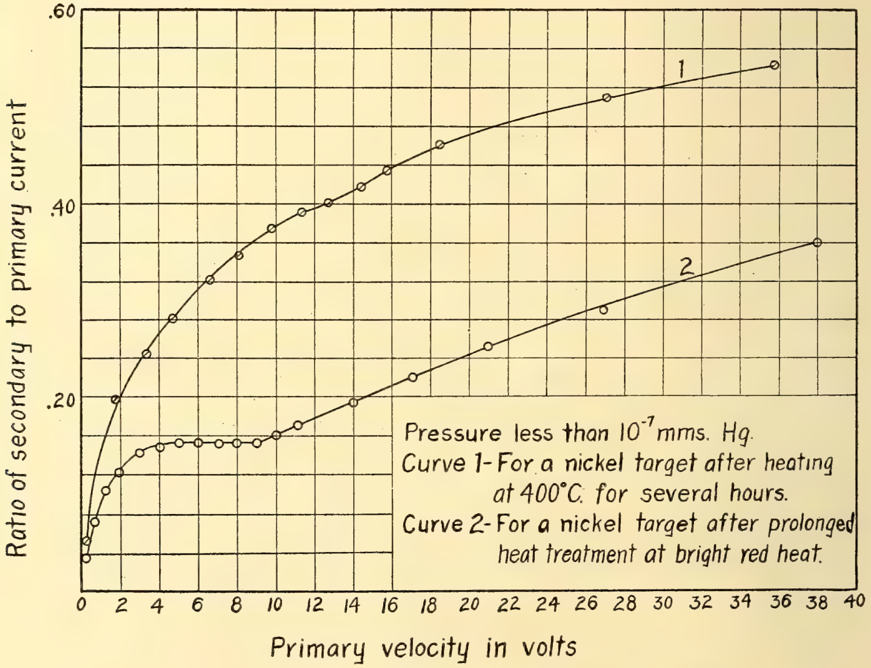

FIG. 1

Measurements of the velocity distribution of the secondary electrons indicate that, below about 9 volts primary velocity, a large percentage of the secondary electrons have velocities nearly equal to the primary velocity and a very small percentage have velocities less than one volt. For higher primary velocities the distribution changes, there being many secondary electrons having low velocities. The percentage of secondary electrons having velocities less than 5 volts, say, increases as the primary velocity increases. There are, however, a few per cent having velocities nearly equal to that of the primary electrons, when the primary velocity is as great as 110 volts.

The following features of the investigation show that a new factor enters into the production of secondary electrons from nickel at about 9 volts: (1) the sudden change in slope of the secondary electron curve at about 9 volts; (2) the difference between the velocity distribution of secondary electrons when the primary velocity is 9 volts or less, and that when the primary velocity is greater than 9 volts. These results lead one to the interpretation that reflection of electrons from nickel occurs for all the primary velocities investigated, the reflection increasing between 0 and 4 volts primary velocity and then remaining constant to 9 volts, where emission begins and increases with increase of primary velocity to at least 260 volts. That there are some high velocity electrons present, when the primary velocity is as great as 110 volts, indicates that reflection continues after emission begins, so that above 9 volts there is both reflection and emission. According to this idea the reflected electrons may be considered to have velocities nearly equal to the primary velocity, i.e., reflection involves a change in direction without much loss in energy, while the emitted electrons have a much smaller velocity.

The above interpretation is further substantiated by a phenomenon which was observed when obtaining the velocity distribution of the primary electrons. It was found that when the primary velocity was less than 9 volts, all of the primary electrons had velocities nearly equal to that corresponding to the applied voltage. For larger primary velocities, however, there were some low velocity electrons present, the number increasing with the primary velocity. This can be explained, in accordance with the previous interpretation, as being due to emission of electrons from the edges of the diaphragms which limit the primary beam. Thus, since below 9 volts no emission occurs there are no low velocity electrons present in the primary beam.

Since the number of secondary electrons obtained at the low velocities was much greater before the continued heating than after, one must conclude that the reflection and emission of electrons from the atoms of gas on a metal surface is greater than from the metal freed from gas.

The method used is particularly well adapted to the extension of the investigation to other metals. Targets of the various metals can be inserted and the characteristics of each studied under the same conditions by moving them successively into the path of the primary electron stream. There is no reason to believe that the secondary electron characteristics should be the same for all metals, and a knowledge of their characteristics in this respect should be of value in connection with the electron theory of metals.

A more detailed account of this work will be published in the *Physical Review* in the near future.

[1] Gehrts, *Ann. Physik,* **36,** 1911, p. 995.

O. von Baeyer, *Physik. Zs.,* **10,** 1909, p. 176; *Deut. Phys. Ges., Verh.,* **10, 24,** 1908, p. 953; *Ber. D. Physik Ges.,* **10,** 1910, p. 96 and p. 953.

N. R. Campbell, *Phil. Mag.,* **22,** 1911, p. 276; **24,** 1912, p. 527; **25,** 1913, p. 803; **28,** 1914, p. 286; **29,** 1915, p. 369.

A. W. Hull, *Physic. Rev.* **7,** 1916, p. 1.

F. Horton and Miss A. C. Davies, *Roy. Soc. Proc. London,* **97A,** 1920, p. 23.

I. G. Barber, *Physic. Rev.,* **17,** 1921, p. 322.

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THE CRYSTAL STRUCTURE OF POTASSIUM

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Communicated June 13, 1922

Potassium, when examined by X-rays at room temperature appears to be amorphous, as previously reported by Hull.[1] But if potassium, cast in a thin-walled cylindrical glass tube, is cooled to —150° C. in a current of air by the method recently employed in getting the crystal structure of mercury,[2] large crystals, giving well marked reflections of the K-radiation of molybdenum, are formed. These crystals are, in fact, so large that it is a matter of some difficulty to obtain enough different orientations in a single sample, and the data are not of great accuracy. Continuous rotation of the sample about the axis of the cylinder so weakens the intensity of the occasional reflections that they can no longer be located with certainty. The best results are obtained by rotating the sample through small angles at such long time-intervals that individual crystals in suitable orientations have long enough to affect the film.

The best fit with the data is furnished by assuming the atoms to be arranged in a body-centered cubic lattice with $a = 5.20 \times 10^{-8}$ cm. and therefore with a density 0.92 gm./cm.[3] This density is higher than the commonly accepted value, 0.862 gm./cm.[3], partly no doubt on account of the low temperature of the crystals, and partly perhaps because the values usually quoted have been obtained for metal not perfectly crystalline, to judge by the X-ray evidence. As regards the purity of the potassium here used, it can only be stated that the original material was purchased as "c. p." from Eimer and Amend, was dried and freed from the usual oily surface layer, and was then melted and double-distilled in vacuo using only pyrex glass. The samples were prepared for me under the supervision of Dr. H. E. Ives from material used by him in the preparation of photoelectric cells. His suggestion that a change in the crystalline condition of the metal at low temperatures might be responsible for certain anomalies in photoelectric emission was the starting point of this research.

The observed crystalline structure does not persist when the temperature is allowed to rise again to about 20° C. No effort has been made to determine the highest temperature at which crystallinity can still be detected, since it seems probable that the loss of structural symmetry is not abrupt but gradual.

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[1] A. W. Hull, *Physic. Rev.*, (2) **10**, 661–696 (Dec., 1917).

[2] L. W. McKeehan and P. P. Cioffi, *Ibid.*, (2) **19**, 444–446 (April, 1922).

PRELIMINARY REPORT ON THE IONIZATION OF POTASSIUM VAPOR BY LIGHT

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Communicated June 13, 1922

As Hughes states in a *Bulletin* of the NATIONAL RESEARCH COUNCIL,[1] experimental results relative to the ionization of metallic vapors by light, have been very meagre, and it is difficult to point to any phenomena which are due, without doubt, to ionization by radiation of optical frequencies. Steubing[2] describes ionization of mercury vapor by light transmitted through fused quartz, but his results seem probably to be explained as ionization in the vapor by collision of photoelectrons from the electrodes. Anderson[3] and Gilbreath[4] have published results obtained in potassium vapor which seem to be due to wall emission. Kunz and Williams,[5] in a brief abstract, state that while caesium was not ionized by radiation of wave-lengths greater than 3130 Å, a marked effect was secured at 2530 Å. No details as to the method have as yet come to the writer's notice.

The equation $Ve = h\nu$, where V is the voltage corresponding to the energy of the electron, e the electronic charge, h the quantum constant, and ν the frequency of radiation, finds application in the field of X-rays, either in calculating the maximum frequency of radiation excited in solids by electrons having energy corresponding to the voltage V, or, conversely, in computing the maximum energy of the secondary electrons liberated from solids by radiation of frequency ν.

Work in the metallic vapors indicates that in the case of the alkaline metals the principal series of doublets is radiated when ionization of the vapor by electronic collision occurs, and the limiting frequency of the series is given by the substitution of the ionizing potential in the above equation. Conversely, when it is desired to ionize a vapor by radiation, thus liberating electrons, it is natural to assume that frequencies must be employed which are equal to, or greater than, the above limiting frequency calculated from the ionizing potential. The results of Kunz and Williams in caesium vapor are at least in accordance with this view. In the case of potassium vapor, the observed ionizing potential is 4.1 volts[6] and the corresponding principal series limit is 2856 Å.[7]

The experiments here described were planned to verify the ionization of metallic vapor by radiation of optical frequencies and to investigate the ionizing power of the radiation as a function of the frequency. The following considerations must be borne in mind in determining the method:

(*a*) Thermelectrons are emitted quite freely by glass and metals in the presence of potassium vapor at temperatures as low as 150°. Also such surfaces in the presence of potassium vapor, or when covered with solid potassium, may give a marked photoelectric emission if there is any scattered light. The procedure must then be arranged so that currents consisting of positive ions, due to ionization, can be distinguished from these electronic currents.

(*b*) Absorption of the radiation by the vapor should be limited, if possible, to the region between the electrodes.

(*c*) Where accelerating fields are used, ionization by collision of electrons with the vapor may occur, and such an effect must be distinguished from ionization by radiation.

Therefore, the following method was adopted. A jet of hot potassium vapor was directed into a cool vacuum chamber and condensed upon the walls, a part of the latter being cooled by means of liquid air. A carefully diaphragmed pencil of light traversed this jet a short distance from the nozzle and then entered a conical glass tube serving as a light trap. A mercury arc in quartz was used as the source of radiation, and absorbing screens could be interposed between the arc and the vacuum chamber. Electrodes were so arranged that the positive ions formed in the jet could be accelerated through a gauze to travel against a suitable retarding field (V_r) to an electrode connected with an electrometer. The energy gained by the positive ions was not due to the full applied accelerating voltage (V_a), since they started from a somewhat extended region between the electrodes. A sliding shutter over the vapor nozzle made it possible to cut off the vapor jet.

Holding the retarding potential constant and varying the accelerating field, observations were made and a curve plotted showing current to the electrometer versus accelerating voltage, when the jet was illuminated by unscreened radiation from the arc. This procedure was repeated with a screen interposed which cut out quite sharply wave-lengths less than 3100 Å. Pairs of such curves were obtained for retarding potentials of 1, 5, and 10 volts, as shown in figure 1.

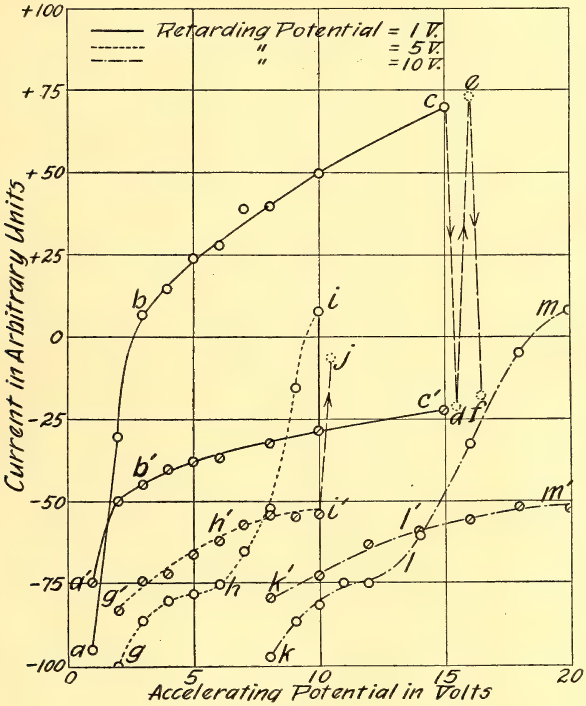


FIG. 1

abc, ghi, klm, were taken with radiation unscreened, minimum wave-length about 1850 Å.; *a'b'c', g'h'i', k'l'm'* were taken with screen, minimum wave-length about 3100 Å. *l'j* gives change when screen is removed. *cd-de* is the change when vapor jet shutter is closed and reopened, and *ef* is due to inserting screen.

The curves taken with the radiation unscreened show negative currents with marked breaks toward positive values at accelerating voltages which depend upon the retarding potentials. The negative currents are due to photoelectric leak from the gauze and walls, which was unavoidable, and also at the lower accelerating voltages to thermelectrons from the nozzle of the jet. The breaks toward positive values resulted when the positive ions, which were produced in the jet by the radiation, acquired sufficient energy in the accelerating field to enable them to reach the electrometer electrode against the retarding potential. When a screen was inserted to cut out wave-lengths less than 3100 Å, the negative leak was reduced by a small amount, while the positive breaks disappeared, showing that the positive currents depend upon the presence of wave-lengths which are less than 3100 Å. It may be suggested that the positive current is due to ionization in the jet by collision of photoelectrons from the gauze, and that the disappearance of that current upon screening out wave-lengths below 3100 Å results from the correspondingly lower initial energy of emission of these photoelectrons. The electrons reach the jet of vapor with their initial emission energy plus at least ½ V_a. Therefore, even though the initial energy were zero in the screened curves, we should have positive ionization by collision when V_a is greater than about 8.2 volts or twice the ionizing potential. And the pairs of curves for V_r greater than 4.1 volts, would show no marked decrease of ionization with screening. On the contrary, the pairs of curves are of the same type for retarding potentials of from 1 to 10 volts, and there is no indication in the screened curves

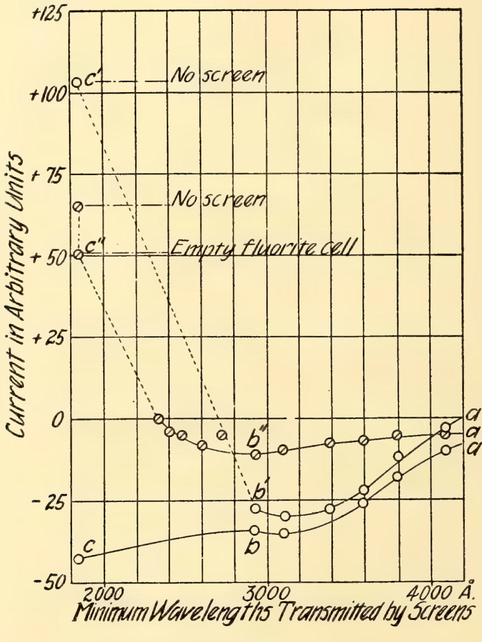


FIG. 2

of ionization by collision for the higher accelerating potentials. Hence we may conclude that the positive ionization is due to radiation between 3100 Å and about 1850 Å (probably the shortest wave-lengths present in the arc spectrum), and also that under the conditions of the experiment ionization by collision was of a lower order of magnitude than that due to radiation.

If the vapor jet is cut off by means of the shutter under conditions as shown in figure 1, approximately the same effect is produced on the current as by inserting the screen in the path of the light. This confirms the view that the negative residual current is due to photoelectric leak, and, furthermore, indicates that the scattered light which produces this leak comes largely from the diaphragming system and is not due to any marked scattering or resonance by the vapor.

In preliminary tests of the ionizing power of the mercury arc spectrum, as a function of wave-length, the retarding and accelerating potentials were set at one and ten volts respectively. Absorbing screens were successively inserted in the beam of radiation, and the observed currents were plotted against the lower transmission limits of the screens, figure 2. Curve *abc* represents the effect of the screens on the photoelectric current from surfaces due to stray light, being taken with no vapor present; $a'b'c'$ and $a''b''c''$ were taken with vapor issuing from the nozzle. No screens for the region below 2900 Å were available at the time *abc* and $a'b'c'$ were taken. These curves show a positive current superposed upon a negative photoelectric surface current. This positive current appears at about 2900 Å and increases quite rapidly as the wave-lengths decrease. The ionization does not seem to be associated with resonance phenomena or the absorption of a single wave-length in the vicinity of 2900 Å. Rather it appears to be a continuous function of the wave-length, starting near the convergence wave-length of the principal series and increasing quite rapidly as the wave-length decreases, giving a continuous absorption of energy below this limit. It is of interest to note, in this connection, the region of continuous absorption described by Wood[8] in the case of sodium vapor which extends from the convergence frequency of the principal doublet series on down in the ultra-violet, thus corresponding with the region of ionization and consequent absorption of radiation observed for potassium.

The above results can be summarized as follows:

It is believed that definite evidence has been obtained of the ionization of metallic vapors by light.

The results indicate that there is a long-wave limit in the case of this ionization which can be calculated from the relation $Ve = h\nu$ by means of the ionizing potential.

The ionizing power of the radiation is a continuous function of the wave-length below this limit, increasing as the wave-length decreases.

The investigation will be continued with the idea of extending the observations to other elements and, also, of getting more accurate measurements of various factors involved.

[1] Hughes, *Bulletin, Nat. Res. Council, Washington,* **2,** 1921 (86).

[2] Steubing, *Physik. Zs.,* **10,** 1909 (787).

[3] Anderson, *Physic. Rev.,* **1,** 1913 (233).

[4] Gilbreath, *Ibid.,* **10,** 1917 (166).

[5] Kunz and Williams, *Ibid.,* **15,** 1920 (550).

[6] Tate & Foote, *Phil. Mag.,* **36,** 1918 (64).

[7] Hughes, *loc. cit.* (168).

[8] Wood, *Phil. Mag.,* **18,** 1909 (531).

THE ABSORPTION OF LIGHT BY SODIUM AND POTASSIUM VAPORS

By George R. Harrison

Department of Physics, Stanford University

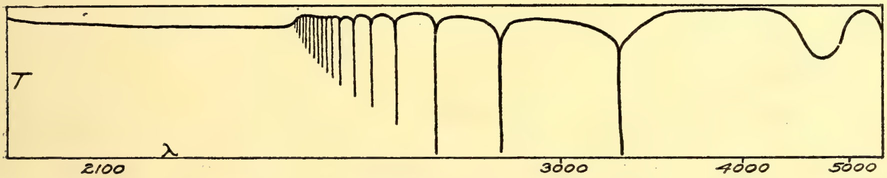
Communicated June 13, 1922

The continuous absorption found at the limit of the principal series in sodium and potassium is of great importance on the Bohr theory of atomic structure, corresponding as it does to the ejection of an electron from the atom. It was first noticed by Wood[1] in sodium, who mentioned that it began in his pictures at the last resolved line, and extended to the extreme ultraviolet. Holtsmark[2] studied it in sodium and found it also in potassium though to a less marked degree.

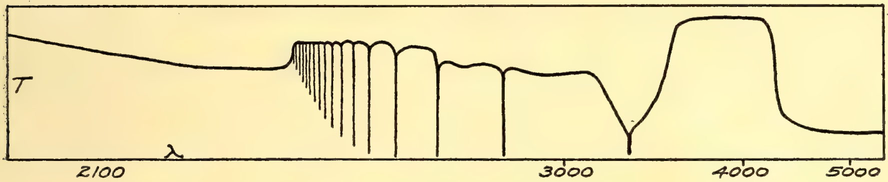
The present work was undertaken to study the relation of the band, line, and continuous absorption in sodium and potassium vapors, to temperature, hydrogen pressure, and vapor saturation. It was thought that some change in absorption might be obtained if the vapor was superheated.

Experimental Arrangements.—The metal was enclosed in iron tubes of 4 cm. diameter and length from 15 to 200 cm., depending upon the type of absorption to be studied. A side arm was welded to the center of the absorption tube, and so arranged that it could be heated separately. Clean sodium was then placed directly in the side arm, and after the main tube had been pumped out and raised to the desired temperature a small quantity of sodium could be distilled into it from the side tube at will. In this way a rough degree of control could be exercised on the saturation of the vapor.

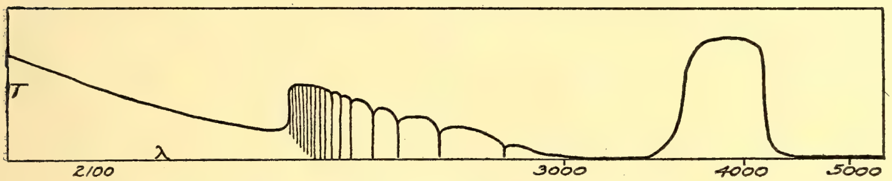
Blast flames were used as a source of heat. The main tube was closed by quartz lenses kept cool with water jackets. To prevent too rapid condensation of the vapor and coating of the windows, slits were placed at each end of the hot column and kept at a red heat. A mercury manometer was kept permanently connected to the main tube, which could be exhausted to a cathode ray vacuum in three minutes with a Gaede cylinder pump. Temperatures were measured with a thermocouple calibrated by the Bureau of Standards. A cadmium spark in air was used as a light source, supplemented by electrically exploded wires of various metals in regions where Cd emission lines interfered. For photographing the transmitted light a Fery spectrograph made by Hilger was used, having a range 2100 to 6000 Angstroms, and a dispersion sufficient to resolve the lines to $n = 32$ in Na.



a. Temp. 390°C. Pressure 2 cm. hydrogen



b. Temp. 475°C. Pressure 3cm. hydrogen



c. Temp. 540°C. Pressure 5cm. hydrogen

FIG. 1

Curves showing the relation of absorption at different wave-lengths to vapor density in saturated sodium vapor, with hot column 30 cm. long. Hydrogen pressures from 1 to 400 mm. had no apparent effect.

Results with Sodium.—The continuous absorption was found to be much more pronounced in certain pictures than in others, but the only determining factor seems to be vapor density. The presence of hydrogen in pressures from one to 400 mm. had no visible effect. The absorption begins at the last resolved line of the principal series, which is about five Angstrom units from the theoretical head. No definite break is observable, which is to be expected, since in the region near the limit there is an integrated intensity of unresolved line absorption and the unabsorbed spaces between.

An important point, which was indicated by all photographs showing the continuous absorption, is that this does not extend undiminished to the extreme ultraviolet. It decreases rapidly with decreasing wave-length; that is, the vapor is much more transparent fifty Angstroms below the head of the series than it is right at the head (2413 Å). In fact, the region at 2200 Å is one of the most transparent in the whole range of wave-lengths studied.

From the analogy to a similar case with X-rays, where continuous absorption decreases as the cube of the wave-length, it would be of interest to know the function determining the falling off of the absorption in the present case. It is expected that in a fuller report on this work to be published elsewhere definite data on this can be given.

In figure 1 a set of curves for sodium is given showing transmission of light as ordinates and wave-lengths as abscissas. These are given to show the manner in which the absorption changes with vapor density throughout the visible and ultraviolet. The curves are to be taken as merely qualitatively correct, since the photometric method used was necessarily rough due to irregularities in the background.

With unsaturated vapor the band absorption is tremendously weakened in comparison to the line and continuous absorption. It is difficult to get pictures which can be compared, since we have no way of determining the relative amounts of sodium present in the absorbing columns in the two cases. The best available method seems to be to take two pictures which show line 2853 (say) absorbed to the same degree in both and use this for a parameter. Two such pictures being chosen it was found that the band absorption was visible around the first four lines in the case of the saturated vapor, and about the D-lines alone, very faintly, in the unsaturated. Potassium is now being tested for a similar effect.

This evidence seems to indicate that the bands may be due to loose molecular aggregates which are broken up when the mean free path becomes greater and the temperature motion more violent. Sodium vapor is supposedly monatomic, but Hackspill's[3] vapor pressure determinations seem to show that loose aggregates may exist, since the results with saturated vapor are not very concordant.

It was noticed that the fluorescence, which was quite marked in saturated vapor, practically disappeared on superheating. Van der Lingen and Wood[4] found a similar effect in mercury vapor. The intimate connection between the mechanisms of fluorescence and band absorption has been definitely established by the work of Wood and others.

Results with Potassium.—This metal showed similar continuous absorption to that of sodium, the saturated vapor only having been studied to date. Apparently Holtsmark's result of less continuous absorption in potassium than in sodium was due merely to difference in vapor density. Curves are given for potassium in figure 2.

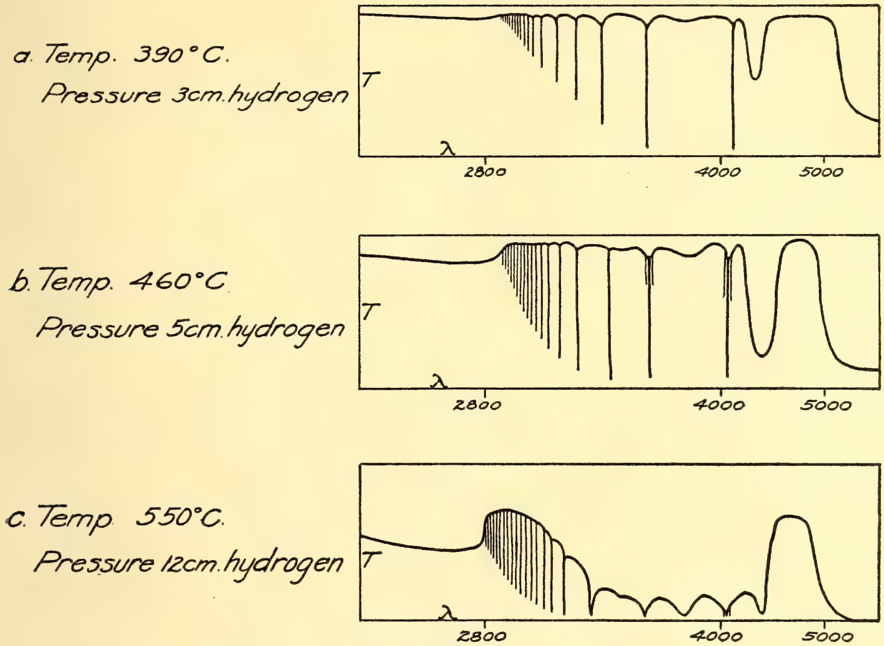


FIG. 2

Potassium

This work is being conducted under the guidance of Dr. D. L. Webster, whose aid is gratefully acknowledged.

[1] Wood, *Phil. Mag.*, **8**, 18, 1909 (530).

[2] Holtsmark, *Physik. Zs.*, **20**, 1919 (88).

[3] Hackspill, *Ann. Chim. Phys.*, **28**, 8, 1914 (613).

[4] Van der Lingen and Wood, *Chicago, Astroph. J.*, **54**, 1921 (149).

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PROCEEDINGS
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Volume 8 SEPTEMBER 15, 1922 Number 9

THE MEANING OF ROTATION IN THE SPECIAL THEORY OF RELATIVITY

By Philip Franklin

Department of Mathematics, Princeton University

Communicated July 27, 1922

The novelty of the special theory of relativity consists in its replacing the equations for translation given by the Newtonian theory by the Lorentz transformations. It is natural to ask what new equations should be used to express a rotation in this theory. Most of the writers on the subject (*e. g.*, A. Einstein, *Leipzig, Ann. Physik*, **49**, 1916, §3; M. V. Laue, *Die Relativitätstheorie*, Vol. 2, 1921, p. 162) have assumed that the Newtonian equations for rotation may be used without change in the new theory. While this is permissible as a "first approximation" for points near the axis of rotation, it is not a satisfactory transformation for the whole of space, since it involves velocities greater than that of light for points sufficiently far from the axis. The object of this note is to present a definition of rotation which is free from this objection, and to deduce some of its properties.

In the Newtonian theory the equations of transformation connecting the coördinates in a system K with those in a system K' moving uniformly with respect to K along the x-axis are:

$$x' = x - vt, y' = y, z' = z, t' = t. \tag{1}$$

For a rotation about the z-axis, we have:

$$x' = x \cos\omega t + y \sin\omega t, y' = - x \sin\omega t + y \cos\omega t, z' = z, t' = t, \tag{2}$$

or if polar coördinates are introduced in the xy plane:

$$\Theta' = \Theta - \omega t, r' = r, z' = z, t' = t. \tag{2a}$$

In the special theory of relativity, in place of (1) we have the Lorentz transformations:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}},\ y' = y,\ z' = z,\ t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}. \qquad (3)$$

To obtain the corresponding generalization of (2a) we shall assume the following properties, which may be used to derive (2a) from (1), as characteristic of uniform rotation.

1. The velocity of a point in K' with respect to the point in K with which it momentarily coincides is independent of the time, and is the same for all points at a given distance from the axis of rotation.

2. For the two concentric circles $r' = r =$ const., the equations of transformation are similar to those for a uniform translation with $r\Theta$ the length of the arc along the circle in place of the linear distance x.

3. The velocity of a point at the distance $r' + \Delta r'$ from the axis with respect to a point at the distance r' from the axis (both in the system K') is a constant (ω) times $\Delta r'$ where $\Delta r'$ is a differential.

From 1 and 2 we find:

$$r'\Theta' = \frac{r\Theta - v(r).t}{\sqrt{1 - v(r)^2/c^2}},\ r' = r,\ z' = z,\ t' = \frac{t - v(r).r.\Theta/c^2}{\sqrt{1 - v(r)^2/c^2}}. \qquad (4)$$

To utilize 3, we recall the equation for compounding velocities, obtained by differentiating (3): $(dx'/dt' = v_{12};\ dx/dt = v_2;\ v = v_1)$

$$v_{12} = \frac{v_2 - v_1}{1 - v_1 v_2/c^2}. \qquad (5)$$

For our case we have $v_2 = v(r + \Delta r)$; $v_1 = v(r)$; $v_{12} = \omega\Delta r$, giving:

$$\omega\Delta r = \frac{v(r + \Delta r) - v(r)}{1 - v(r + \Delta r)\, v(r)/c^2}, \qquad (6)$$

or, in the limit,

$$\omega \frac{dr}{dv} = \frac{1}{1 - v^2/c^2}, \qquad (7)$$

and on integration, taking the constant to make $v(0) = 0$:

$$v = c \tanh \omega\, r/c \qquad (8)$$

By inserting this value in (4), we obtain as our final equations:

$$\begin{aligned} \Theta' &= \Theta \cosh \omega r/c - ct\,\frac{\sinh \omega r/c}{r},\ r' = r, \\ z' &= z,\ t' = t \cosh \omega r/c - r\Theta\,\frac{\sinh \omega r/c}{c}, \end{aligned} \qquad (9)$$

From the form of these equations we see that the totality of transformations corresponding to them forms a group; since the identity is given by setting $\omega = 0$, the inverse of the transformation $\omega = \omega'$ by setting $\omega = -\omega_1$, and the resulting of the two transformations $\omega = \omega_1$ and $\omega = \omega_2$ by setting $\omega = \omega_1 + \omega_2$. We further note that r is restricted to quantities small compared with c, we may replace cosh $\omega r/c$ by unity and sinh $\omega r/c$ by $\omega r/c$, and on neglecting the term in r^2/c^2 we obtain (2*a*). This justifies the use of (2*a*) as a first approximation to obtain results for actual rotating systems, for which of course r/c is negligibly small. (Cf. the discussion of Sagnac's experiment given by M. V. Laue, *Relativitätstheorie*, Vol. 1, 1919, p. 125 and by L. Silberstein in *J. Optical Soc. Amer.*, **5**, 1921, p. 291 f.)

While the relation between the primed and unprimed coördinates, given by (9) is formally symmetrical, the corresponding spaces are not interchangeable. In fact, if the unprimed space is a "stationary" system the square of the arc element of the space-time continuum is given by:

$$d\sigma^2 - c^2\,dt^2 \tag{10}$$

where $d\sigma$ is the arc element for three-dimensional Euclidean space, and consequently in our coördinates by

$$dz^2 + dr^2 + r^2\,d\Theta^2 - c^2\,dt^2. \tag{11}$$

The corresponding element in the rotating, primed system is obtained by solving (9) and substituting the values in (11). This, of course, leaves the four-dimensional space unchanged; but the three dimensional section obtained by putting t' equal to a constant is no longer the arc element of Euclidean space, but is given by the expression (in which primes are dropped):

$$\begin{aligned} & dz^2 + r^2 d\Theta^2 + 2r\,dr\,d\Theta\left(\omega t - \frac{ct}{r}\sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c} - \Theta\sinh^2\frac{\omega r}{c}\right) \\ & \quad + dr^2\left(1 - \frac{r^2\omega^2\Theta^2}{c^2} + \omega^2 t^2 + \frac{c^2t^2}{r^2}\sinh^2\frac{\omega r}{c} - \Theta^2\sinh^2\frac{\omega r}{c}\right. \qquad (12) \\ & \left. - 4\,\omega\Theta t\sinh^2\frac{\omega r}{c} - 2\,\frac{ct^2\omega}{r}\sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c} - 2\frac{r\Theta^2\omega}{c}\sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c}\right). \end{aligned}$$

To prove that the space with the above element is non-Euclidean, we have merely to show that its Gaussian curvature is not zero. We only calculate it for the case $t' = 0$. For this special case of (12) the curvature is:

$$R = \frac{\frac{\omega^2}{c^2}\left(1 + 2\Theta^2\left(\frac{\omega r}{c} + \sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c}\right)\left(\frac{\omega r}{c}\sinh\frac{\omega r}{c} - \cosh\frac{\omega r}{c}\right)\left(\frac{\cosh\frac{\omega r}{c}}{\frac{\omega r}{c}}\right)\right)}{\left[1 - \Theta^2\left(\frac{\omega r}{c} + \sinh\frac{\omega r}{c}\cosh\frac{\omega r}{c}\right)^2\right]^2} \qquad (13)$$

It appears from the calculations just made that the spacial geometry for the rotating system depends on the time and space coördinates of the point considered, and this is at first sight in contradiction with our ordinary ideas of rotating systems. The explanation of this fact is that for a system rotating with respect to a "stationary" system there is no separation into space and time which stands out as the natural one for the entire rotating system, and consequently for this system spacial geometry is only defined when our coördinates have been selected. Since the equations (9) depend on coördinates so chosen that when $t' = 0$, the points on the radius vector $\Theta^1 = 0$ coincide with those on the radius vector $\Theta = 0$, at $t = 0$ in the stationary system, it is based on coördinates "natural" to these space-time points; in the sense that they are those of a stationary system coinciding with the rotating one at the space-time points considered. It will be noticed that for these points the curvature given by (13) is ω^2/c^2. This shows that the curvature of the spacial cross-section at any space-time point in the coördinates "natural" to this point is constant; it is the square of the angular velocity of rotation in radians per light-second.

THE ENERGY LOSSES ACCOMPANYING IONIZATION AND RESONANCE IN MERCURY VAPOR

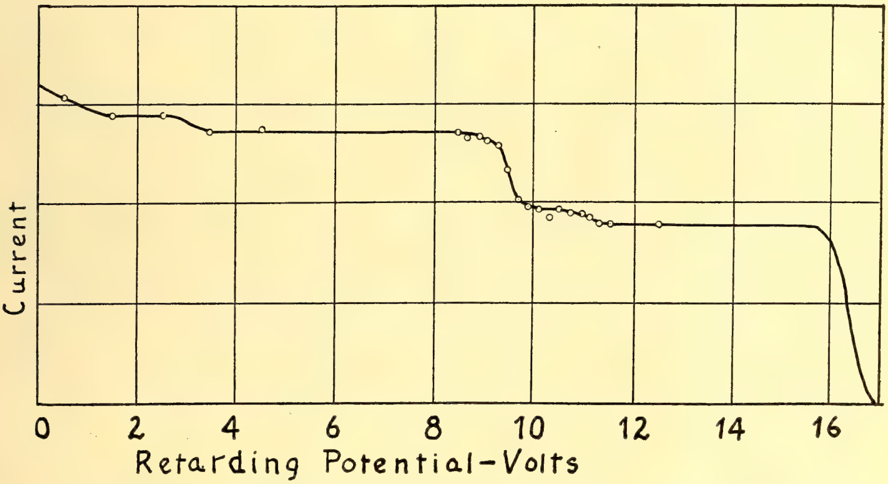
By John A. Eldridge

Department of Physics, University of Wisconsin

Communicated June 28, 1922

Much light has in recent years been thrown upon the constitution of matter, and upon the validity of the Bohr theory of atomic structure by the study of resonating and ionizing collisions of electrons in vapors and gases. There are, however, still a great many questions of a fundamental nature in regard to such phenomena which cannot be answered by the experimental methods which have hitherto been employed for the determination of the resonating and ionizing potentials. It seemed desirable to use a new type of tube, with which the actual distribution of velocities of the electrons after their collisions with the vapor molecules could be determined.

Instead of the grid employed in the usual form of tube two diaphragms, placed one behind the other and each pierced with a single small hole, were used. These diaphragms divided the tube into two regions: an upper one in which any desired pressure of mercury vapor could be maintained and in which the collisions to be studied took place; and a lower region from which mercury was removed by liquid air and in which the electron velocities were measured. In the upper region the electrons were emitted from an oxide coated cathode and accelerated by a grid which was maintained at any desired potential. Passing through the grid, the electrons entered a constant potential region and after a large number of collisions with the mercury vapor, some passed through the two small holes into the lower, mercury free region and to the receiving electrode. To this electrode was applied a variable retarding potential and by measuring the current as a function of this retarding voltage the velocity distribution of the electrons was determined.



In the curve shown the current is plotted against the retarding potential for electrons which had, before collision, velocities ranging between 15.8 and 16.8 volts. We find in the curves drops beginning at 0, 3, 9.1, 10.9 and 15.8 volts, respectively. The drop at 15.8 volts represents those electrons which after collision retained all of their energy. The other drops in the current are due to groups of electrons which have lost respectively all, 13.4 (i.e. 2 × 6.7), 6.7 and 4.9 volts of energy which they had before collision. No drop is seen at 5.4 volts which would correspond to the loss of 10.4 volts which is normally assumed to take place at ionization.

From curves of such nature for different velocities of impact, it has been possible to determine more surely than heretofore the energy losses at electron impact. The outstanding results of the investigation are:

1. The discovery by Mohler, Foote and Meggers of a resonance potential involving an energy loss of 6.7 volts was confirmed. The work from which these investigators were able to deduce the existence of this critical potential seemed subject to some very serious difficulties in interpretation. These difficulties have been cleared up. The 6.7 volt type of collision does not occur for electrons with energies less than 8.5 volts, but at voltages above the ionization point it is the most important type of resonating collision.

2. Indications were also found that a collision involving an energy loss of about 5.7 volts also occurs in mercury. This is in agreement with the fact that absorption lines have been found by previous investigators in the mercury spectrum in a region which through the quantum relation corresponds to this potential.

3. Little has been hitherto known about the nature of the ionizing collision. The present method has cast some light on this matter. The results obtained indicate that at such a collision the impinging electron loses all of its energy, and also that the electron which is the product of the ionization leaves the parent atom with negligible energy. This conclusion is at variance with the usual conception of the ionization process.

THE CRYSTAL STRUCTURE OF BERYLLIUM AND OF BERYLLIUM OXIDE

By L. W. McKeehan

Research Laboratories of the American Telephone and Telegraph Company and the Western Electric Company, Incorporated

Communicated July 17, 1922

The structure of the lighter elements and of their compounds should, on account of their greater simplicity, be of especial value in deciding what rôle the so-called valence electrons play in connecting atoms. Beryllium (Be; atomic number 4) is the lightest metal the crystal structure of which has not hitherto been determined by X-ray methods. Its accepted atomic weight is $w = 9.10$, although a recent determination,[1] apparently of high accuracy, gives $w = 9.018$ which is in much better agreement with the evidence indicating that the element consists of a single atomic species.[2]

Through the kindness of one of my colleagues, Dr. H. E. Ives, a small supply of metallic beryllium, recently prepared for him in this laboratory by Mr. G. O. Smith, was made available to me for this study. The method of preparation was by the electrolysis of $2BeF_2.NaF$ with from 7 to 15 amperes at 80 volts. This is a modification of the method of Fichter and Jablczynski[3] who used a lower voltage. The product of the process is a fine gray powder interspersed with thin flakes up to half a millimeter in greatest dimensions, which show good metallic lustre. The principal impurity to be expected is beryllium oxide (BeO), and this should not exceed in amount a few per cent by weight.

A thin-walled pyrex glass tube about one millimeter in diameter was packed loosely with the beryllium powder, the looseness of packing being intended to reduce the chance of anisotropy.[4] This sample was not thick enough to cast a very sharply defined shadow on the film, a circumstance which somewhat reduces the accuracy of the results.[5] A water-cooled molybdenum target, bombarded by about 30 milliamperes of electrons at about 30,000 volts, and provided with ZrO_2 filters[6] gave an intense source of X-rays, the photographically important wave-lengths being[7] 0.71212×10^{-8} cm. (K α_2) and 0.70783×10^{-8} cm. (K α_1). The radius of the photographic film was 20.36 cm. The time of exposure was about 18 hours. In taking the first photograph the cylindrical sample was kept stationary; in taking the second photograph the sample was rotated about its axis through a small angle at intervals of one hour. The two photographs were quite similar in general appearance, but the second had smoother lines in its pattern, so that it alone was used in the final measurements and calculations.

In order to eliminate from consideration any lines certainly due to the presence of BeO a sample of that substance in the form of a fine powder was examined in the same way. The results are presented in Table I, and show that the diffracting centers lie at the points of two symmetrically interpenetrating hexagonal space lattices, each with the axial ratio $c/a = 1.63$ and with $a = 2.696 \times 10^{-8}$ cm. There are three faint lines which must be attributed to the presence of impurities. Assuming that one molecule, BeO, is associated with each lattice point and taking[8] $N = 6.0594 \times 10^{23}$, $w = 9.018$, this gives a computed density 2.99 gm./cm^3. Both structure and density are in agreement with published values,[9] although a recent paper by Gerlach[10] contains a very different result. A careful examination of his data, however, shows that the resolving power of his apparatus was not high, and that all the lines which he reported as intense are within a short distance of positions accountable for by the structure here proposed.

Comparing the pattern obtained for Be with that obtained for BeO it was found that the first three lines in the former and possibly also the fourth were attributable to the oxide and its principal impurity. A brief search by graphical methods[11] showed that the fundamental space lattice was hexagonal, and that the fourth line of pattern might be retained. A more exact analysis gives the results presented in Table II, and shows that the diffracting points form two symmetrically interpenetrating hexagonal lattices, each with the axial ratio $c/a = 1.58$ and with $a = 2.283 \times 10^{-8}$ cm. No lines are left unaccounted for this structure, which is consistent with the hexagonal structure ($c/a = 1.5802$) determined by optical methods.[12] Assuming that one atom is associated with each lattice point, and taking N and w as given above, this gives a computed density 1.827 gm./cm^3., or about one per cent less than the best previous values.[13] The relative positions of the two component space lattices along the hexagonal axis are not determined with any accuracy, the intensities being very roughly estimated and not particularly significant.

Comparing the proposed structure for Be with those already reported for elements in the same column of the periodic table, it is clear that it belongs rather with Mg, Zn, and Cd than with Ca, Sr, and Ba, the first three being hexagonal, all with c/a greater than 1.58 and increasing with the atomic

TABLE I—Be

ESTIMATED INTENSITY	MEASURED $2r\Theta$ IN CM.	COMPUTED LOG (10^8d)	ATTRIBUTED TO[1]	COMPUTED[2] LOG (10^8a)
2	4.69	0.4911	X[3]	
6	6.60	0.3437	BeO	
1	6.97	0.3201	BeO	
4	7.44	0.2921	(10.0)	0.3546
2	8.05	0.2583	(00.2)	0.3607
10	8.37	0.2416	(10.1)	0.3611
2	11.02	0.1244	(10.2)	0.3587
4	12.89	0.0583	(11.0)	0.3593
2	14.44	0.0108	(10.3)	0.3576
	[4]		(20.0)	
	[4]		(11.2)	
2	15.55	$\bar{1}.9801$	(20.1)	0.3593
	[5]		(00.4)	
	[5]		(20.2)	
1	18.33	.9128	(10.4)	0.3572
			Mean	0.3586
			Result a =	2.283×10^{-8} cm.

[1] All planes theoretically capable of reflection within the range of the tabulated values of $2r\Theta$ are included.

[2] Computed for $c/a = 1.58$.

[3] X = impurity in BeO.

[4] Diffuse band not resolved. The theoretical range in $2r\Theta$ is only 0.58 cm.

[5] Not observed with certainty.

number, whereas Ca, the only member of the second three which has been analyzed, has cubic symmetry.[14] The remaining element in this column, Hg, has rhombohedral symmetry.[15]

The oxide, BeO, finds an analog in ZnO, all the other monoxides in the column having cubic symmetry,[16] except HgO which forms monoclinic crystals.[17] The structure of ZnO has been reported by W. L. Bragg[18] to be hexagonal close-packing of molecules with $c/a = 1.608$ and $a = 3.22 \times 10^{-8}$ cm. Recent data obtained by Hedvall[19] but not reduced by him is in good agreement with this structure. Bragg regards the two hexagonal lattices of the proposed arrangement as determining the positions of the zinc atom-centers, and suggests that the oxygen atom-centers lie at the points of two similar lattices derived from these by displacing both of them along the hexagonal axis by about $3c/8$. The same type of structure is at

TABLE II—BeO

ESTIMATED INTENSITY	MEASURED 2rθ IN CM.	COMPUTED LOG (10^8d)	ATTRIBUTED TO[1]	COMPUTED[2] LOG (10^8a)
2	4.69	0.4911	Y[3]	
1	5.43	0.4277	X	
8	6.23	0.3685	(10.0)	0.4310
6	6.67	0.3392	(00.2)	0.4301
10	7.09	0.3128	(10.1)	0.4297
1	7.57	0.2847	X	
3	9.15	0.2035	(10.2)	0.4311
8	10.90	0.1290	(11.0)	0.4300
8	11.89	0.0923	(10.3)	0.4308
	[4]		(20.0)	
6	12.86	0.0593	(11.2)	0.4303
	[4]		(20.1)	
	[4]		(00.4)	
	[4]		(20.2)	
	[4]		(10.4)	
2	16.40	$\bar{1}.9582$	(20.3)	0.4293
1	16.90	$\bar{1}.9459$	(21.0)	0.4309
1	17.29	$\bar{1}.9366$	(21.1)	0.4303
	[4]		(11.4)	
	[5]		(10.5)	
2	18.34	$\bar{1}.9126$	(21.2)	0.4304
	[4]		(20.4)	
1	19.35	$\bar{1}.8911$	(30.0)	0.4305
2	20.04	$\bar{1}.8770$	(21.3)	0.4298
			Mean	0.43032
			Result a =	2.696×10^{-8} cm.

[1] All planes theoretically capable of reflection within the range of the tabulated values of 2rθ are included.

[2] Computed for $c/a = 1.63$.

[3] X = impurity.

[4] Not observed with certainty.

[5] Forms with next an irresolvable doublet ($\Delta s = 0.01$ cm.).

least consistent with the data for BeO. In both cases there should be different properties in the two directions of the hexagonal axis.

The fact that c/a is greater for Zn than for Be, but less for ZnO than BeO is hard to explain unless we assume a considerable rearrangement of electrons to take place during the formation of one of the oxides. It is interesting to note that another substance which possesses this structure is ice,[20] which tempts one to regard frozen water as an oxide of the formula $(OH_4)O$, the radical OH_4 being divalent, just as NH_4 is univalent and CH_4 is saturated.

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SOME CASES OF NERVE-DEAFNESS AND THEIR BEARING ON RESONANCE THEORIES OF AUDITION

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1. *Introduction.*—The present paper deals with certain cases of nerve-deafness and their bearing on the mechanical resonance theories of tone perception. The paper is thus a continuation of a recent one published in the February issue, 1922, of the *Physical Review* and like the earlier

one, is a result of further coöperation with some of the ear specialists of Chicago.

It is generally accepted among medical men that the purpose of an end-organ is to lower the threshold of perception of certain sensations. In the case of the ear, the complicated internal-ear structure is not only for the purpose of lowering the threshold of audition, but, according to all the mechanical resonance theories of pitch perception, it is also for the purpose of making possible the phenomenon of this perception. Data which have now been collected show with much certainty that these resonance theories are untenable and it is the purpose of the present paper to present some of these data. The results seem to show clearly that the sole purpose of the mechanical structure of the internal-ear is to lower the threshold of audition and that the power of pitch perception is a property of the nerves including their endings. According to this view, then, the power of pitch perception is in no way associated with the actual mechanical structure of the cochlea.

2. *Experimental Results.*—The details of the method of testing were described in the paper to which reference has already been made and these need not be discussed in this place. Many cases of nerve-deafness have been examined but only those that have a direct bearing upon the purpose in mind will be presented.

a. One of the most interesting set of curves for nerve-deafness is that shown in figure 1. This set is for a person 45 years of age. The upper curve was taken in December, 1920. The lower curve for the same ear was taken in March, 1922, so an interval of sixteen months had elapsed. The curves show that the ear functioned in a normal manner up to 1000 d. v. So, as pointed out in the previous paper, we know that no deficiency in hearing exists because of middle-ear trouble. During the period of sixteen months the impairment of hearing has become quite marked; requiring about 665,000 times as much current through the receiver for the threshold of audition at 3000 d. v. as is required by a normal ear. The minimum of the earlier curve has shifted from 3300 d. v. to 3000 d. v. during the interval. The range in frequency for which a deficiency in hearing exists includes lower pitches than were included in the first test, and, moreover, over the affected range the impairment is much more marked than indicated by the first curve. It is almost incredible that a person could be as deaf as indicated by these curves at some pitches and still have normal hearing at other frequencies.

In the case of the patient under discussion, near the threshold of audition at the higher pitches, say in the region of 3000 or 4000 d. v., the tones would break through, so to speak; they would be heard as pure tones with much intensity. A moment later it seemed as though the supply of nerve energy had been exhausted and the sensation of a pure tone would change to a blurred noise of just audible intensity. When the tone was heard no noise sensation was present and when the noise was heard no tone was observed. So, it would seem that the same nerves were responding under both conditions and that a certain minimum amount of nerve energy was necessary for tone perception. It is quite improbable that such a variation as this could be due to the mechanical condition of the internal-ear but it would appear most reasonable to consider it due to a condition of the auditory nerve system whose impairment, without doubt, was responsible for the deficiency in hearing. Even though there was such an impairment in hearing due to the condition of the nerves and their endings the patient was able to recognize all the pitches when the tones were sufficiently intense.

b. In figure 2 is shown a group of six curves for the right and left ears of three partially deaf and dumb persons who had congenital deafness. Impaired hearing of this nature is caused, according to medical findings, by the incomplete or mal-development of the internal-ear system. These six curves are of much value from the theoretical viewpoint as will be seen later.

Curves 1 and 2 are for the two ears of a young man 25 years of age. Curves 3 and 4 are for those of a young girl 18 years of age and Curves 5 and 6 are for the two ears of a boy 10 years old. The first two curves are of the same general shape; both have their maximum depression at 3300 d. v., but the degree of hearing for the two ears is quite different. Curves 3 and 4 are also of the same general shape and the same is true of Curves 5 and 6. All three of these patients recognized all the pitches, the tones sounded of the same pitch when listened to by the two ears (that is there was present no indication of diplacusis for these cases) and the pitch discriminating power was apparently normal. Therefore, the auditory nerve systems for these three patients responded in a normal manner in regard to pitch perception. The sound intensity, of course, needed to stimulate the nerves in order to produce the sensation of sound, was far more than required for normal hearing; this being due, no doubt, to the structural condition of the cochlea.

Although these three persons were nominally deaf and dumb, all three of them recognized the difference between the fourteen fundamental vowel sounds when they were intoned in the patients ears. In the case of the girl the vowel sounds were repeated correctly by her. She was able to recognize and repeat, but with difficulty, spoken words and simple sentences. The results were not as satisfactory in the case of the boy. Nevertheless, he was able to repeat the vowel sounds with sufficient clearness to make it certain that he was able to distinguish the difference between these sounds.

c. In figure 3 are given two curves for patients each of whom were considered by the otologist totally deaf in one ear. The threshold curves of audition showed that both patients were able to hear all the tones at sound intensities of not unusually large magnitudes, and furthermore both patients had about the same impairment of hearing. They were also able to recognize perfectly all the tones of the various pitches at the increased intensities. Care was taken to make sure that the patients were hearing with the tested ears and not with the opposite ears. In the case of the patient represented by Curve 1 the opposite ear possessed normal hearing at 1000 d. v. and 4000 d. v. and it was assumed that the opposite ear was normal. The opposite ear, however, in the case of the patient represented by Curve 2 was quite defective in hearing. The possibility existed

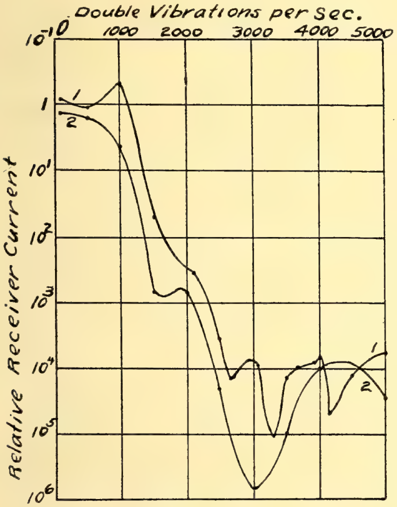

FIG. 1

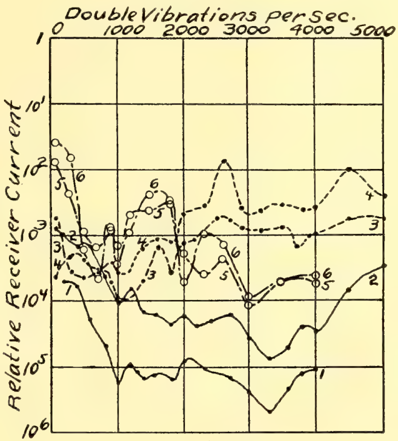

FIG. 2

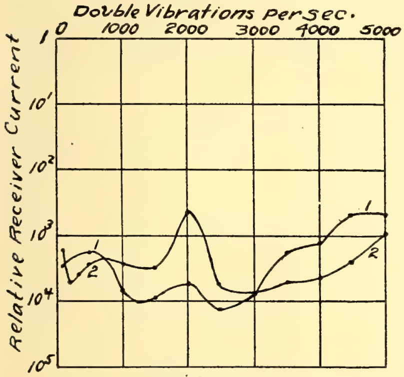

FIG. 3

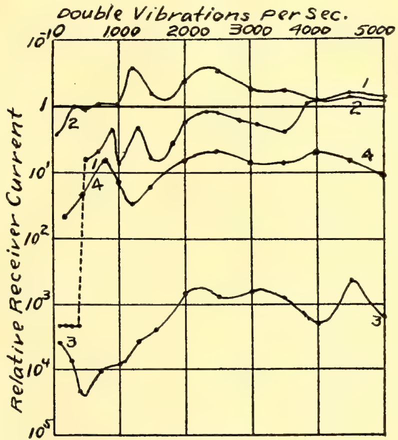

FIG. 4

that these patients were hearing in these opposite ears and that they were totally deaf in the other ears, as diagnosed. However, the moistened finger was placed in the ear which was not being tested and in the case of both patients the sensation of sound disappeared when the receiver was removed a cm. or so from the ear. They were able to hear when the receiver was brought close to but not in contact with the ear and they were unable to hear when the receiver was placed in contact with the bones of the head. There was also present the possibility that these two patients were not hearing the tones in the tested ears but through the eustachian tubes and the mouth to the opposite ears. Sufficient data are at hand, it is believed, to negate such a conclusion as this. It was apparent, then, that both patients heard by air conduction with the ears diagnosed as totally deaf. The diagnosis, of course, rested upon the fact that the structure of the cochlea was known to have been completely destroyed by disease and an operation in one of the cases. In spite of this destruction, however, both patients heard all the tones up to 5000 d. v. and their auditory nerve systems responded in a normal manner.

d. Two additional cases of true nerve deafness will be cited. The curves for these are shown in figure 4. Curves 1 and 2 are for a girl 20 years old whose ears were in perfect condition at the time of the test. She was affected on one side only by paralysis of the facial, auditory and vestibular nerves. These three nerves join one another at a definite and known location in the head. Since all three of the nerves were simultaneously affected and since they recovered at the same time, the exact location of the lesion was definitely known to the otologist. Curve 1 for this case shows an abrupt break between 400 and 500 d. v. At pitches below the break all the tones were heard as some sort of a noise corresponding to the noise referred to in the discussion of the curves in figure 1. The noise rather than a tone was present even when two or three thousand times as much current was passing through the receiver as was required for audition by a normal ear. Above the break in the curve the tones were perceived as pure ones at intensities only slightly greater than were required by normal ears. Thus, the nerves, stimulated in a perfectly normal manner, would not transmit the sensation of a pure tone at any of the pitches below the break but would transmit the sensations undistorted at all pitches above this point. A lesion, then, in the nerve proper may prevent the formation and transmission to the brain of a stimulus corresponding to a pure tone.

This patient was first tested on February 4, 1921, and was again tested on March 7, 1921. During this period she had recovered completely and her hearing as shown by Curve 2 in figure 4 was equivalent to or better than normal. So, with the disappearance of the cause of the paralysis there was an immediate return to normal hearing and pitch perception.

The concluding case to be cited was also one of pure nerve lesion due to pressure on the auditory nerve brought about by the growth of a cerebral tumor. Curve 3, figure 4, shows the relative degree of hearing possessed by the patient's left ear. The right ear was not greatly affected; as shown by Curve 4 it required for minimum audition a relative receiver current of 48.5 at 200 d. v. and improved with increasing pitch, requiring a relative receiver current of 11.5 at 5000 d. v. The curve (4) for the right ear, however, is almost exactly of the same shape as that (3) for the left ear. The lesion, then, had affected both ears alike qualitatively. In the case of this patient it has not been found possible to secure additional curves.

These two cases of true nerve deafness show the maximum depression in hearing at the lower frequencies, whereas for most cases of what is termed nerve deafness caused by lesions within the internal-ear the greatest depressions occur at the higher pitches.

Many additional curves have been taken on patients with nerve deafness, but for the purpose of the present paper it is believed that the cases cited are sufficient to establish the object in view.

3. *Theoretical Bearing of the Cases Cited on the Mechanical Theories of Tone Perception.*—As stated in the introduction, it is believed that the data presented herewith show conclusively that the sole purpose of the mechanical structure of the internal-ear is simply to lower the threshold of audition and that mechanical resonance of this structure is not in the slightest degree responsible for the phenomenon of tone perception. The group of curves shown in figure 3, for patients known to have internal-ears completely modified, showed normal pitch perception but greatly reduced sensitivity. If mechanical resonance of this structure were responsible for tone perception, then this perception would have been completely destroyed in these two cases. This sense remained unmodified, but the greatly decreased sensitivity of hearing would suggest conclusively that the internal-ear structure is for the prime purpose of lowering the threshold of audibility.

This view is much strengthened by the group of six curves in figure 2 for deaf-mutes all of whom showed simply reduced sensitivity of hearing without any apparent modification of the sense of tone perception. Among medical men it is considered that congenital deafness of this sort means that the cochlea is only partially developed. Accepting this view as even partially correct, we are forced to the conclusion that all theories of audition which are based on mechanical resonance of the internal-ear are untenable.

The data for Curve 1 are the same as given in figure 17 of the paper published in the February 1922 issue of the *Physical Review,* p. 88. The argument against the mechanical resonance theories as stated in this earlier paper, pp. 95–96, can be repeated here with even more decisiveness than was given there because the patient has become more deaf at the higher pitches as shown by Curve 2, figure 1, since the first curve was taken.

It would seem, therefore, even in the case of the complete modification of the internal-ear mechanism that a person can hear in a normal manner, provided the tones are sufficiently intense and provided there has not been a complete destruction of the nerve endings or of the nerves themselves in which case we should expect complete deafness. Only one such case has as yet been found and in this case the patient was unable to hear at any pitch in either ear with as much as 5×10^6 times as much current passing through the receiver as was required by a normal ear.

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THE QUANTITATIVE INFLUENCES OF CERTAIN FACTORS INVOLVED IN THE PRODUCTION OF CYANOSIS

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Lundsgaard has shown that the appearance of cyanosis depends on the mean concentration of reduced hemoglobin (C) in the capillary blood. This concentration he estimated as

$$C = \frac{A + V}{2} \qquad (1)$$

where A is the arterial, V the venous concentration of reduced hemoglobin. The effect of certain physiological factors contributing to C is estimated as follows. We let T represent the total hemoglobin concentration in the blood, l the fraction of total hemoglobin passing in reduced form through the aerated parts of the lungs, D the concentration of reduced hemoglobin formed by deoxygenation of the blood as it passes from arteries to veins through the tissue capillaries. In certain pathological conditions a fraction of the venous blood reaches the arteries without traversing aerated parts of the lungs and therefore carries reduced venous hemoglobin directly into the arterial blood. This fraction we designate as α. A, T, and C have the significance indicated above.

When $\alpha = O$, only aerated blood entering the arteries, $A = l\,T$. When however, α has a positive value, the reduced hemoglobin of the arterial blood represents the sum of that in the fraction α, of venous blood entering the arteries, plus that of the fraction $(1 - \alpha)$ of arterial blood from the lungs. In such a case

$$A = (1 - \alpha)\, l\, T + \alpha V \tag{2}$$

$$\alpha = \frac{A - l\,T}{V - l\,T} \tag{3}$$

Substituting $V - D$ for A in (3) and solving for V,

$$V = 1\,T + \frac{D}{l - \alpha} \tag{4}$$

Substituting $A + D$ for V in (3) and solving for A,

$$A = l\,T + \frac{\alpha D}{1 - \alpha} \tag{5}$$

We now substitute for A and V in (1) their values as expressed in (4) and (5), and obtain

$$C = l\,T + \frac{(1 + \alpha)D}{2(1 - \alpha)} \tag{6}$$

which indicates the effects of the four factors, l, T, α, and D on C.

An interesting point brought out by (6), and in accord with observation, is that other factors normal about 40 per cent of venous blood may be mixed with arterial ($\alpha = 0.4$) before the reduced hemoglobin concentration in the latter becomes great enough to cause cyanosis.

The effects on C of separate variations of l, T, D, and α are indicated by partial differentiation of (6).

$$\partial C = T\,\partial l \tag{7}$$

$$\partial C = l\,\partial T \tag{8}$$

$$\partial C = \frac{D}{(1 - \alpha)^2}\,\partial \alpha \tag{9}$$

$$\partial C = \frac{(1 + \alpha)}{2(1 - \alpha)}\,\partial D \tag{10}$$

While the reduced hemoglobin concentration in the capillary blood may be accepted as the cause of cyanosis, there are various other factors which modify the resulting coloration. Such are local skin vascularity, pigmentation, thickness of epidermis; and also the fact that the mean capillary content of reduced homoglobin $\frac{1}{2}(A + V)$ only approximately represents the average content. With changing conditions the latter may instead of being midway between venous and arterial, approximate either more nearly than the other. The effect of these modifying factors is to cause the value of C at which cyanosis becomes perceptible to vary from 4 to 6 gms. of reduced hemoglobin per 100 cc. of blood, and perhaps sometimes even more widely, although it appears usually to lie near 5.

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NEW PROPERTIES OF ALL REAL FUNCTIONS

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Communicated June 10, 1922

In a former paper[1] the author communicated a number of properties of every real function $f(x)$, which were stated in terms of the successive saltus functions associated with a given function. The present results do not involve these saltus functions and are direct qualifications of $f(x)$. Since $f(x)$ is entirely unrestricted, except, of course, that it is defined—even this restriction may be partially dispensed with—and therefore finite for every real x, these qualifications are consequences of nothing else than that $f(x)$ is a function. A new light is thus thrown upon the nature of a function.

The new properties are of two kinds, descriptive and metric; the former are concerned with density and the latter with measure (Lebesgue).

For the sake of greater concreteness, we discuss, for the most part, planar sets and real, single-valued functions of two real variables.

1. *Descriptive Properties.*—We say that a planar set S is an "I-region" (= open set) if every point of S is an inner point of S. We deal with binary relations $\Re$ between I-regions and points. $I \Re P$ shall mean that the I-region I has the relation $\Re$ to the point P. The relation $\Re$ is said to be "closed" if the relationships $I \Re P_n$ and $\lim\limits_{n \rightarrow \infty} P_n = P$ imply $I \Re P$. By a "neighborhood" of a point P, we understand an I-region containing P; by a "partial neighborhood" of P, an I-region of which P is an inner or a boundary point.

We have the following

Lemma I. If $\Re$ *is a closed relation, then the points for which* (*a*) $N \Re P$ *for every neighborhood* N *of* P, *and* (*b*) *a partial neighborhood* $N_{<}$ *exists such that* $\overline{N_{<} \Re P}$ (i.e., $N_{<} \Re P$ is false) *constitute a non-dense* (i.e., nowhere dense) *set.*

An important example of a closed relation appears in connection with a function. Let $z = f(x, y)$ be any given one-valued function of the two real variables x and y, and let f be defined for the entire XY plane. Let $\mathfrak{R} = \mathfrak{R}_{r_1r_2}$, where $r_1 < r_2$ are two real numbers, be defined as follows: $I\mathfrak{R}_{r_1r_2}P$ if and only if an infinite sequence of points P_n of the I-region I exists such that $\lim\limits_{n \to \infty} P_n = P$, $\lim\limits_{n \to \infty} f(P_n)$ exists, and $r_1 \leqq \lim\limits_{n \to \infty} f(P_n) \leqq r_2$; here P, $P_n =$ (x, y), (x_n, y_n) and $f(P), f(P_n) = f(x, y), f(x_n, y_n)$.

Definition. The function $f(x, y)$ is said to be "densely approached at the point (ξ, η)" or in other words, the point $(\xi, \eta, f(\xi, \eta))$ of the "surface" $z = f(x, y)$ is said to be densely approached, if for every positive ϵ there exists a planar neighborhood N of (ξ, η) such that the points of N for which $|f(x, y) - f(\xi, \eta)| < \epsilon$ form a dense set in N.

Definition. An "exhaustible" set (= set of first category, according to Baire) is the sum of $\aleph_0$ non-dense sets; a "residual" set, the complement (with respect to the XY plane) of an exhaustible set.[2]

By the aid of Lemma I and the closed relation $\mathfrak{R}_{r_1r_2}$, we prove

Theorem I. For every real function $f(x, y)$ whatsoever, the points of the surface $z = f(x, y)$ that are densely approached form a residual set. Conversely, given any residual set R whatsoever, a function $f(x, y)$ exists that is densely approached at and only at the points of R.

The following definition of dense approach is equivalent: The function $f(x, y)$ is said to be densely approached at $P = (\xi, \eta)$ if, for every partial neighborhood $N_<$ of P, the set of points of the surface $z = f(x, y)$ corresponding to the points of $N_<$ has $(\xi, \eta, f(\xi, \eta))$ as a limit point. By the use of this definition, we get the following theorem, which is equivalent to Theorem I, but which shows better, perhaps, the remarkable degree of "microscopic symmetry" an unconditioned function possesses.

Theorem I′. With every function $f(x, y)$ whatsoever, there is associated a residual set R—dependent on f—of the XY plane such that if $P = (\xi, \eta)$ is a point of R and $N_<$, a partial neighborhood of P, then $(\xi, \eta, f(\xi, \eta))$ is a limit point of the set of points $(x, y, f(x, y))$ for which (x, y) is in $N_{\triangleleft}$.

Definition. The function f is said to be "inexhaustibly approached" at the point P if every neighborhood of P contains, for every $\epsilon > 0$, an inexhaustible set of points—i.e., a set that is not exhaustible—at which f differs from $f(P)$ by less than ϵ.

If M is any planar set, we use, in connection with approach, the expression "via M" to designate that (x, y) is restricted to range in M. Thus "f is inexhaustibly approached at P via M" means that for every neighborhood N of P and every $\epsilon > 0$, the set MN, which is the aggregate of points common to M and N, contains an inexhaustible set of points at which f differs from $f(P)$ by less than ϵ.

We have the following

Theorem II. For every function $f(x, y)$, there exists in the XY plane a *residual set R—dependent on f—such that if P is a point of R, and* $N_<$, *a partial neighborhood of P, the function f is inexhaustibly, and therefore densely approached at P via* $RN_<$.

With the aid of Theorem II we prove

Theorem III. With every function $f(x, y)$ *there is associated—not uniquely, however—a dense set D of the XY plane, such that f is continuous if* (x, y) *ranges over D.*

2. *Generalizations.*—The considerations and theorems of Section 1 apply not only to functions of n real variables, but to every space S that satisfies the following four conditions:

(1) S is metric;[3] that is to say, with every pair of elements P and Q of S there is associated a non-negative, real number $\overline{PQ}$ (Fréchet's écart) in such a way that if P, Q and R are three elements of S, then

(*a*) $\overline{PQ} = \overline{QP}$;

(*b*) $\overline{PQ} = 0$, when and only when $P = Q$; and

(*c*) $\overline{PQ} + \overline{QR} \geqq \overline{PQ}$.

(2) S is complete (vollständig);[4] that is to say, if $\{P_1, P_2, \ldots, P_n \ldots\}$ is a "regular" sequence of elements of S—in other words, for every $\epsilon > 0$ there exists an integer n_ϵ such that $\overline{P_\lambda P_\mu} < \epsilon$ for $\lambda > n_\epsilon$ and $\mu > n_\epsilon$—there exists a limit element P (i.e., an element P with the property $\lim\limits_{n \to \infty} \overline{P_n P} = 0$).

(3) S contains a denumerable subset that is dense in S.

(4) S has no isolated points.

We may thus state the following theorem—the definitions of the terms for S may be obtained after slight and evident modifications from those for the plane.

Theorem IV. Let S be any complete, metric space containing a dense, denumerable subset and without isolated points; and $f(P)$, *any real function defined for the elements P of S. Then there exists a residual set R, such that if P is a point of R, and* $N_<$ *a partial neighborhood of P, the function f is inexhaustibly and therefore densely approached at P via* $RN_<$. *Also there exists a dense subset D of S such that* $f(P)$ *is continuous if P ranges over D.*

As particular examples of a complete, metric space, with a dense, denumerable subset and without isolated points, we mention:

(*a*) Euclidean n-space where the écart between two points is the euclidean distance between them.

(*b*) A perfect subset of euclidean space.

(*c*) Hilbert space, that is, the ensemble of sequences $(x_1, x_2, \ldots x_n, \ldots)$ of real numbers with convergent $\sum\limits_{n=1}^{n=\infty} x_n^2$. The écart between two "points" $(x_1, x_2, \ldots x_n, \ldots)$ and $(y_1, y_2, \ldots y_2, \ldots)$ is defined to be $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots}$.

(*d*) Function space: S consists of all real, continuous functions $f(x)$

defined for $0 \leqq x \leqq 1$. The écart between $f_1(x)$ and $f_2(x)$ is defined to be $\max|f_1(x) - f_2(x)|$.

The assumption that f is single-valued may also be dropped without invalidating Theorems I and II—Theorem III, of course implies single-valuedness by its very nature. We thus get the following generalization of Theorem I (and a similar one for Theorem II).

Theorem I $^{(a)}$. *Let $f(x, y)$ be any real function defined for the entire XY plane and taking at every point at least one value—the number of values may change, however, from point to point and vary from 1 to c, the cardinal number of the continuum. Then the points (x, y) such that every surface point $(x, y, f(x, y))$ is densely approached by the surface $z = f(x, y)$ constitute a residual set.*

3. *Metric Properties.*—As in the case of Section 1, we discuss in this section planar sets and functions of two variables.

Let S be any planar set; P, a point of S; C_r, a circle with P as center and r as radius; $m(C_r)$, the area of C_r, and $m_e(SC_r)$, the exterior Lebesgue measure of the portion of S in C_r. Then if

$$\lim_{r \to 0} \frac{m_e(SC_r)}{m(C_r)}$$

exists and is equal to k, we say that the "exterior metric density" of S at the point P is k. We have the following

Theorem V.[5] *Let S be any planar set. Then the points of S at which the exterior metric density of S is* $\neq 1$—i.e., the points where the exterior metric density either does not exist or does exist and is less than 1—*constitute a set of zero measure (Lebesgue).*

Definition. $N_<$ is said to be a "*non-vanishing* partial neighborhood of P," if the exterior metric density of $N_<$ at P is $\neq 0$. We have the following lemma, which corresponds to Lemma I for the descriptive properties:

Lemma II. Let $\mathfrak{R}$ be a closed relation as in Lemma I. The points P for which (a) $N \mathfrak{R} P$ for every neighborhood of P and (b) a non-vanishing partial neighborhood $N_<$ exists such that $\overline{N_< \mathfrak{R} P}$ (i.e., *$N_< \mathfrak{R} P$ is false*) *constitute a set of zero measure.*

By the aid of this lemma we prove

Theorem VI. Let $f(x, y)$ be any real, one-valued function defined in the entire plane. Then there exists in the XY plane a set Z—dependent on f—of measure zero, such that if (a) (x, y) is any point of the XY plane not belonging to Z; (b) $N_<$, any non-vanishing partial neighborhood of (x, y); and (c) S, any sphere with $(x, y, f(x, y))$ as center; then there is at least one point of the surface $z = f(x, y)$ lying in the sphere S and having as projection upon the XY plane a point in $N_<$.

This theorem becomes false if we omit the restriction that the partial neighborhood $N_<$ shall be non-vanishing.

By means of Lemma II, we prove also

Theorem VII. The set of points P at which f is inexhaustibly approached and for which a non-vanishing partial neighborhood exists via which f is exhaustibly approached constitute a set of measure zero.

Definition. f is said to be "neglectably approached at the point $P \equiv (\xi, \eta)$ via M" if a sphere S exists with $(\xi, \eta, f(\xi, \eta))$ as center, such that the projection upon the XY plane of the set of points common to S and the surface $z = f(x, y)$ has a set of measure zero in common with M.

The following theorem generalizes Theorem VI.

Theorem VIII. Let $z = f(x, y)$ be any real, one-valued function defined in the XY plane. Then the points P of the XY plane that possess a non-vanishing partial neighborhood via which f is neglectably approached at P constitute a set of zero measure.

Definition. f is "quasi-continuous"[6] at P if for every ϵ the set of points Q for which $|f(Q) - f(P)| < \epsilon$ has 1 as exterior metric density at P.

We have the following theorem, which generalizes theorem *VIII*.

Theorem IX. f is quasi-continuous except at the points of a set of measure zero.

Concluding Remarks.—As in the case of the descriptive properties, the metric theorems may be extended to many-valued functions. Theorem VIII, for example, when thus generalized reads as follows: *Let $z = f(x, y)$ be any real, single- or many-valued function defined in the entire XY plane. Then the points (x, y) of the XY plane for which a surface point $(x, y, f(x, y))$ and a non-vanishing partial neighborhood $N_<$ exist such that $(x, y, f(x, y))$ is neglectably approached via $N_<$ constitute a set of zero measure.*

The metric properties hold for functions of a single variable and, in general, for functions of n variables. Extension to $\aleph_0$ space, to function space and to more general spaces would require a satisfactory definition of measure for such spaces;[7] it is not our purpose in this paper to enter upon such questions.

Instead of projecting the surface points of $z = f(x, y)$ upon the XY plane, we may project them upon the X-axis and thus obtain other properties. For example, let us define the relationship $\mathfrak{R}_{r_1r_2r_3r_4}$—between I-regions and points of the X-axis—as follows: $I \; \mathfrak{R}_{r_1r_2r_3r_4} \; \xi$ if the surface points having x-coördinates in I have a limit point in the rectangle $x = \xi$ $r_1 \leqq y \leqq r_2$, $r_3 \leqq z \leqq r_4$. $\mathfrak{R}_{r_1r_2r_3r_4}$ is closed. By applying Lemma II to this closed relation, we obtain the following result: *Let $z = f(x, y)$ be any single- or many-valued function defined in the entire XY plane. Let ξ be a point of the X-axis of the following character: a surface point (ξ, η, ζ) and a partial non-vanishing (linear) neighborhood $N_<$ of ξ exist such that (ξ, η, ζ) is not a limit point of surface points having x-coördinates in $N_<$. The totality of points ξ is of measure zero.*

Similar results may be obtained for other metric properties and also in the case of the descriptive properties. In the case of a function of n-variables, we may project upon an $(n - 1)$ space, an $(n - 2)$ space, etc.

The proofs of the preceding theorems are contained in a paper that has been offered to the *Transactions of the American Mathematical Society.*

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[1] *Ann. Math. Princeton,* **18,** 1917 (147).

[2] For the terminology cf. Denjoy, *J. Math., Paris,* ser. 7, **1,** 1915 (122–125).

[3] Cf. for example, Fréchet, *Rend. Circ. Math. Palermo,* **22,** 1907, p. 1; and Hausdorff, *Grundzüge der Mengenlehre,* 1914, p. 211.

[4] Hausdorff, l. c., p. 315.

[5] For the case where only measurable sets are admitted cf., for example, de la Vallée Poussin, *Cours d' Analyse,* **2,** 1912, p. 114. For the linear case of general (not necessarily measurable) sets cf. Blumberg, *Bull. Amer. Math. Soc.,* **25,** 1919 (350).

[6] Cf. Denjoy, *Bull. Soc. Math. France,* **43,** 1915 (165).

[7] In this connection cf. Gâteaux, *Ibid.,* **47,** 1919 (47).

GENERALIZED LIMITS IN GENERAL ANALYSIS

By Charles N. Moore

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Communicated July 15, 1922

It is well known that to each of the various methods for summing divergent series there corresponds an analogous method for summing divergent integrals. It is readily seen that similar methods may be used for obtaining types of generalized derivatives of a function at a point where the ordinary derivative fails to exist. Likewise, in any other case in Analysis where we wish to associate a limit with a variable that oscillates, we will naturally be led to make use of methods that have been tried out in the case of divergent series.

It would be manifestly poor economy of time and thought to elaborate for each of these special theories such fundamental results as are common to them all, if these results can be obtained in one central theory that includes all the others. According to a principle of generalization formulated by E. H. Moore and stated by him on several occasions,[1] the existence of such a general theory is implied by the analogies found among the various special theories. It is natural to designate this general theory as the theory of generalized limits in General Analysis.

It is the purpose of the present communication to illustrate the nature and usefulness of this theory by outlining the proof of a theorem in it which is a generalization of one of the important theorems in the theory of divergent series. This latter theorem is the Knopp-Schnee-Ford theorem[2] with regard to the equivalence of the Cesàro and Hölder means of order k

for summing divergent series. Our genreal theorem includes as special cases this theorem, the analogous theorem of Landau[3] for divergent integrals, and a further new theorem with regard to the equivalence of generalized derivatives of the Cesàro and Hölder type. It can also be extended to the case of multiple limits, so that it includes new theorems, analogous to those mentioned, with regard to multiple series, multiple integrals, and partial derivatives. As this latter extension involves more complicated formulas, it will not be dealt with in the present communication.

Following the terminology introduced by E. H. Moore,[4] we indicate the basis of our general theory as follows:

$$(\mathfrak{A}\,;\mathfrak{P}\,;\mathfrak{S}\,;\mathfrak{G}^{\text{on }\mathfrak{S}\text{ to }\mathfrak{A}}\,;\mathfrak{H}^{\text{on }\mathfrak{S}\text{ to }\mathfrak{A}}\,;\mathfrak{F}^{\text{on }\mathfrak{S}\text{ to }\mathfrak{A}}\,;\varphi_0\,;J^{\text{on }\mathfrak{G}\text{ to }\mathfrak{H}\,.\text{ on }\mathfrak{H}\text{ to }\mathfrak{F}}),$$

where $\mathfrak{A} \equiv [a]$ denotes the class of all real numbers a, $\mathfrak{P} \equiv [p]$ denotes a class of elements p, and $\mathfrak{S} \equiv [\sigma]$ denotes a class of sets σ of elements p of the range $\mathfrak{P}$; $\mathfrak{G} \equiv [\gamma]$, $\mathfrak{H} \equiv [\eta]$, and $\mathfrak{F} \equiv [\varphi]$ are three classes of functions γ, η, and φ, respectively, on $\mathfrak{S}$ to $\mathfrak{A}$ (we shall restrict ourselves throughout to the consideration of single-valued functions); φ_0 is a special function φ of the class $\mathfrak{F}$; and J is a function on $\mathfrak{G}$ to $\mathfrak{H}$ and on $\mathfrak{H}$ to $\mathfrak{F}$, that is, a functional transformation turning a function of the class $\mathfrak{G}$ into a function of the class $\mathfrak{H}$ or a function of the class $\mathfrak{H}$ into a function of the class $\mathfrak{F}$, denoted by $J\gamma$ or $J\eta$.

In order to make clear the relationship of our general theorem to the two special theorems referred to above, we will indicate here what the general basis reduces to in the particular instances III and IV:

$$\mathfrak{P}^{\text{III}} \equiv [\text{all } n = 1, 2, \ldots\ldots\ldots];\ \ \mathfrak{S} \equiv [\sigma_n \equiv (1, 2, \ldots\ n) \,|\, n];$$
$$\mathfrak{G} \equiv \mathfrak{H} \equiv \mathfrak{F} \equiv [\text{all } \gamma, \eta, \varphi \text{ on } \mathfrak{S} \text{ to } \mathfrak{A}];$$
$$\varphi_0(\sigma_n) = n\ \ (n);\ \ (J\gamma)\ (\sigma_n) = \gamma(\sigma_1) + \gamma(\sigma_1) + \ \ldots\ldots\ + \gamma(\sigma_n)\ \ (n).$$

$$\mathfrak{P}^{\text{IV}} \equiv [\text{all } a > 0];\ \ \mathfrak{S} \equiv [\sigma \equiv (\text{all } x,\ 0 < x < a)\ \ (a > 0)];$$
$$\mathfrak{G} \equiv [\text{all functions that are finite and integrable}$$
$$\text{(Lebesgue) on every finite interval } (0, a)];$$
$$\mathfrak{H} \equiv [\text{all } \eta = \textstyle\int_0^x \gamma \,|\, x > 0)];\ \ \mathfrak{F} \equiv [\text{all } \varphi = (\textstyle\int_0^x \eta \,|\, x > 0)];$$
$$\varphi_0(\sigma_a) = a\ (a);\ (J\gamma)\ (\sigma_a) = \textstyle\int_0^a \gamma\ (\gamma a);\ (J\eta)\ (\sigma_a) = \textstyle\int_0^a \eta\ (\eta a).$$

We next proceed to make certain postulates with regard to the nature of the elements in our basis, readily seen to be verified in the specific instances cited. Thus we require the class $\mathfrak{G}$ to have the linear property (L) as defined by E. H. Moore,[4] and the property (P) defined by

$$(P) \qquad\qquad \gamma_1 \cdot \gamma_2 \cdot \supset \cdot\, \gamma_1\gamma_2\mathfrak{G},$$

the arguments of the two γ's being the same. It will then follow that $\gamma_P = \gamma_1 \cdot \gamma_2 \cdot \cdots \cdot \gamma_n$ is of the class $\mathfrak{G}$. We require the class $\mathfrak{H}$ to have the properties (L) and (P), and the further property of being a sub-class of the

$\mathfrak{G}$, which property we shall denote by S_G. We require the class $\mathfrak{F}$ to have the properties (L) and (P), and the further property of being a sub-class of the class $\mathfrak{H}$, which property we shall designate as S_H. Hence is J also on $\mathfrak{F}$ to $\mathfrak{F}$.

We require the operation J to have the linear property (L) as defined by E. H. Moore, and the properties M_1 and M_2, defined as follows:

$$(M_1) \quad a_1<\gamma_1<a_2\cdot 0 \leqq \gamma_2\cdot\gamma_3 = \gamma_1\gamma_2\cdot \supset \cdot a_1\,(J\gamma_2)\,(\sigma) \leqq (J\gamma_3)\,(\sigma) \leqq a_2(J\gamma_2)\,(\sigma)\,(\sigma),$$

$$\begin{aligned}(M_2) \quad a_1<\gamma_1<a_2\cdot 0 \leqq \gamma_2\cdot\gamma_3 = \gamma_1\gamma_2\cdot\sigma'' \supset \sigma'\cdot \supset \cdot\ & a_1[(J\gamma_2)\,(\sigma'')-(J\gamma_2)\,(\sigma')] \\ & \leqq (J\gamma_3)\,(\sigma'')-(J\gamma_3)\,(\sigma') \\ & \leqq a_2[(J\gamma_2)\,(\sigma'')-(J\gamma_2)\,(\sigma')].\end{aligned}$$

For the explanation of the other postulates we need certain auxiliary definitions. By the notation $\sigma' < \sigma''$, $\sigma''>\sigma'$, we shall mean that σ'' contains all the elements of σ' and at least one element not found in σ'. For a given θ on $\mathfrak{S}$ to $\mathfrak{A}$, a given a, and a given σ' such that there exists $\sigma<\sigma'$, we shall write

$$\lim_{\sigma|\sigma<\sigma'} \theta(\sigma) = a$$

in the case that, corresponding to an arbitrary positive number e, there exists a $\sigma_e<\sigma'$ such that for every σ having the property $\sigma_e \subset \sigma<\sigma'$, $|\theta(\sigma) - a|<e$, or in symbols

$$e \supset :: \exists\ \sigma_e<\sigma', \sigma_e \subset \sigma<\sigma'\cdot \supset \cdot |\theta(\sigma)-a| <e.$$

This latter definition is based on the fundamental definition of limit in General Analysis given by E. H. Moore in the paper in these PROCEEDINGS referred to in the first foot-note.

We now postulate for the class $\mathfrak{S}$ the following properties:

$(\boldsymbol{\cup})$[5] Either $\sigma'\cdot \supset \cdot \exists\ \boldsymbol{\cup}[\sigma<\sigma']^{\mathfrak{S}}$ or $\exists\ \sigma_0, \sigma'>\sigma_0\cdot \supset \cdot \exists \boldsymbol{\cup}[\sigma<\sigma']^{\mathfrak{S}}$,
$(\boldsymbol{\cap})$[5] $\sigma'\cdot \supset \cdot \exists\, \boldsymbol{\cap}[\sigma>\sigma']$.

In the typical instances in view in the formation of this general theory, of the two alternatives in $(\boldsymbol{\cup})$ one holds and the other does not hold; however it is not assumed that this disjunction between the two alternatives shall be presupposed. In order to avoid notational reference to these alternatives, it is convenient to introduce a property—of sets σ of $\mathfrak{S}$; if the first alternative holds, every σ of $\mathfrak{S}$ has the property—, in notation $\bar{\sigma}$; if the first alternative does not hold, the sets $\bar{\sigma}$ are the sets $\sigma>\sigma_0$; further for brevity a property $\cdot$ (the negation of $-$) is used; thus every σ is a $\bar{\sigma}$ or a $\dot{\sigma}$.

We define

$$\begin{aligned}&\sigma'_{-1} \equiv \boldsymbol{\cup}\,[\sigma<\sigma'] \quad (\bar{\sigma}),\\ &\sigma'_1 \equiv \boldsymbol{\cap}\,[\sigma>\sigma'],\ \sigma'_{n+1} \equiv \boldsymbol{\cap}\,[\sigma>\sigma'_n] \quad (\sigma\,|\,n=1,2,3\cdots),\\ &\bar{\varphi}(\sigma) \equiv \varphi(\sigma_{-1}) \qquad (\bar{\sigma}),\quad \bar{\varphi}(\sigma) \equiv 0 \quad (\dot{\sigma}),\end{aligned}$$

and with regard to the functions $\bar{\varphi}$ we postulate

$$(F) \quad \text{all functions } \bar{\varphi} \text{ are of the class } \mathfrak{F}.$$

We also define the notation

$$(D\theta)\ (\sigma) = \alpha(\sigma) \tag{1}$$

with regard to every θ, α on $\mathfrak{S}$ to $\mathfrak{A}$, to mean that

$$\lim_{\sigma \mid \sigma < \sigma'} \frac{\theta(\sigma') - \theta(\sigma)}{\varphi_0(\sigma') - \varphi_0(\sigma)} = \alpha(\sigma')\ \ (\bar{\sigma}),\ \theta(\sigma) = \alpha(\sigma)\ \ (\sigma).$$

We further postulate as to the class $\mathfrak{S}$

$$\sigma' \cdot \supset \cdot \exists\ (\sigma_1')_{-1} = \sigma'\ \ (\sigma'), \tag{R}$$

for the class $\mathfrak{H}$

$$\eta \cdot \sigma' \cdot \supset \cdot \exists \lim_{\sigma \mid \sigma < \sigma'} \eta(\sigma) = \bar{\eta}(\sigma')\ \ (\bar{\sigma}'), \tag{C}$$

and for the class $\mathfrak{F}$

$$\exists\ D\varphi \equiv [(D\varphi)\ (\sigma)\ |\sigma]^{\mathfrak{H}}. \tag{\Delta}$$

For the operation J we postulate

$$\eta \cdot \supset \cdot (D(J\eta))\ (\sigma) = \eta(\sigma)\ \ (\sigma), \tag{I_D}$$

$$\varphi \cdot \supset \cdot (J(D\varphi))\ (\sigma) = \varphi(\sigma)\ \ (\sigma). \tag{I_J}$$

We wish finally to postulate certain properties of the special function φ_0. We need first to introduce a further definition. We shall mean by the notation

$$\lim_{\sigma} \eta(\sigma) = \infty$$

that for every positive e there exists a σ_e (dependent on e) such that for every $\sigma > \sigma_\epsilon$, $\eta(\sigma) > e$, or in symbols

$$e : \supset : \exists \sigma_e \ni \sigma > \sigma_e \cdot \supset \cdot \eta(\sigma) > e.$$

This definition also is based on E. H. Moore's definition of a limit referred to above. We also introduce for the sake of brevity the following notations

$$\varphi_{0n}(\sigma) = \varphi_0(\sigma) \cdot \varphi_0(\sigma_1) \cdot \varphi_0(\sigma_2) \cdot \cdots \cdot \varphi_0(\sigma_{n-1})\ (n > 1),\ \varphi_{01}(\sigma) = \varphi_0(\sigma), \tag{2}$$

$$\psi_n(\sigma_1) = \frac{1}{\varphi_{0,n-1}\ (\sigma_1)}\ (n > 1). \tag{3}$$

We now postulate for φ_0 that

$$\varphi_0 \text{ is a real and positive function of } \sigma, \tag{I}$$

(II) $$\lim_{\sigma | \sigma < \sigma_n}{}' \, [\varphi_0(\sigma_n') - \varphi_0(\sigma)] = 1 \quad (\sigma' n),$$

(III) $$\frac{1}{\varphi_{0n}(\sigma)} (J^n \eta)(\sigma) \text{ as function of } \sigma \text{ is of the class } \mathfrak{F}\ (n),$$

the symbol $J^n\eta$ indicating that the operation J has been repeated n times

(IV) $$\varphi_0(\sigma_n) \text{ as a function of } \sigma \text{ is of the class } \mathfrak{F}\ (n),$$

(V) $$\lim_{\sigma} \varphi_{0n}(\sigma) = \infty \quad (n)$$

(VI) $$(D\varphi_{0n})(\sigma) = n\varphi_{0,n-1}(\sigma) > 0 \quad (\sigma, n > 1), \quad (D\varphi_{01})(\sigma) = 1 \quad (\sigma),$$

(VII) $$(\mathrm{D}\psi_n)(\sigma_1) = -\frac{n-1}{\varphi_{0n}(\sigma)} \quad (\sigma, n > 1).$$

We have then as the foundation of our theory:

$$\Sigma \equiv (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{S};\ \mathfrak{G}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A},\ LP};\ \mathfrak{H}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}.\ LPS_G C};$$
$$\mathfrak{F}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}\ LPS_H \Delta};\ \varphi_0{}^{\mathfrak{F}.\ \text{I II III IV V VI VII}};$$
$$J^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H}.\ \text{on } \mathfrak{H} \text{ to } \mathfrak{F}.\ LM_1M_2I_DI_J}).$$

We can now readily prove the two following properties of the operations D and J as applied to the class $\mathfrak{F}$:

(4) $$(D[\varphi_1\varphi_2])(\sigma) = \varphi_2(\sigma)(D\varphi_1)(\sigma) + \varphi_1(\sigma)(D\varphi_2)(\sigma) \quad (\sigma),$$

(5) $$(J[\varphi_2(D\varphi_1)])(\sigma) = \varphi_1(\sigma)\varphi_2(\sigma) - (J[\varphi_1(D\varphi_2)])(\sigma) \quad (\sigma).$$

We next formulate a further limit definition that is also based on the definition of E. H. Moore to which reference has already been made. We shall say that $\eta(\sigma)$ approaches a limit a as to σ if, corresponding to every positive e, we can find a σ_e such that for $\sigma > \sigma_e$ we have $|\eta(\sigma) - a| < e$, or in symbols

(6) $$\lim_{\sigma} \eta(\sigma) = \mathrm{a} :\cdot \equiv :\cdot e : \supset : \exists \sigma_e \ni \sigma > \sigma_e \cdot \supset \cdot |\eta(\sigma) - a| < e.$$

We are now ready to define the two generalized limits with which we are concerned. Given any function $\eta(\sigma)$, we set

(7) $$(C_n\eta)(\sigma) = [n!/\varphi_{0n}(\sigma)](J^n\eta)(\sigma) \equiv \frac{n!}{\varphi_0(\sigma)\ \varphi_0(_1) \cdots \varphi_0(\sigma_{n-1})} (J^n\eta)(\sigma),$$

(8) $$(M\eta)(\sigma) = [1/\varphi_0(\sigma)](J\eta)(\sigma),$$

(9) $$(H_n\eta)(\sigma) = (M^n\eta)(\sigma),$$

where C and H are used, as is customary, to connote Cesàro and Hölder. If for a fixed n $\lim_{\sigma} (C_n\eta)(\sigma)$ exists, we define this limit as the generalized limit of type (Cn) for $\eta(\sigma)$. If $\lim_{\sigma} (H_n\eta)(\sigma)$ exists, we define this limit as the generalized limit of type (Hn) for $\eta(\sigma)$.

We can now prove the equivalence of these two generalized limits by a method similar to that devised by Schur[6] for proving the special theorem in the case of series. The proof breaks up into three lemmas and a theorem.

LEMMA 1. *If we represent by E the identical functional operation, $E\theta = \theta(\theta)$, we have the identity*

$$((\tfrac{n-1}{n} M + \tfrac{1}{n} E)\ (C_n\eta))(\sigma) = (M(C_{n-1}\eta))\ (\sigma). \tag{10}$$

LEMMA 2. $\exists \lim\limits_{\sigma} \varphi(\sigma) = a \cdot |\varphi(\sigma)| < a_1\ (\sigma) \cdot \supset \cdot$

$$\exists \lim_{\sigma}\ [\varphi_{0n}(\sigma)]^{-1}(\,J\varphi_n)(\sigma) = \frac{a}{n},$$

where

$$\varphi_n(\sigma) = \varphi_{0,n-1}(\sigma)\varphi(\sigma)\ (n \geqq 2), \quad \varphi_1(\sigma) = \varphi(\sigma),$$

$\varphi_{0n}\ (\sigma)$ *being defined by* (2).

LEMMA 3. *If we set*

$$\omega(\sigma) = \tfrac{1}{n}\,\varphi(\sigma) + \tfrac{n-1}{n} \cdot \frac{1}{\varphi_0(\sigma)}\,(J\varphi)(\sigma),$$

$$\exists \lim_{\sigma} \omega\ (\sigma) = a \cdot \supset \cdot \exists \lim_{\sigma} \varphi(\sigma) = a.$$

Theorem. $\exists \lim\limits_{\sigma}\ (C_n\eta)(\sigma) = a \cdot \sim \cdot \exists \lim\limits_{\sigma}\ (H_n\eta)(\sigma) = a.$

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[3] E. Landau, "Die Identität des Cesàroschen und Hölderschen Grenzwertes für Integrale," *Leipzig Berichte,* **65,** 1913, pp. 131–138.

[4] Cf. the first three references in the first foot-note; also E. H. Moore, "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations," *Proc. Fifth Inter. Congr Math.* (Cambridge, 1912), **1,** 1913, pp. 230–255.

[5] For the definition of these symbols see page 150 of the monograph referred to in the first footnote.

[6] I. Schur, "Über die Äquivalenz der Cesàroschen und Hölderschen Mittelwerte," *Math. Annalen,* **74,** 1913, pp. 447–458.

A PRELIMINARY NOTE ON TETRAMITUS, A STAGE IN THE LIFE CYCLE OF A COPROZOIC AMOEBA

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In 1920 a study of the Entamoeba of the rat was begun and in a culture on artificial medium a number of Coprozoic amoeba appeared, one of which exhibited a flagellate phase which seems to be identical with Tetramitus rostratus Perty.[1]

The appearance of flagellate phases in cultures of Coprozoic amoeba (Whitmore,[2] etc.) as well as of soil amoeba (Kofoid,[3] Wilson,[4] etc.) has been recorded a number of times, but the flagellates which appeared were relatively simple in organization and very transitory in occurrence. Furthermore, some of the simpler flagellates have been observed (Pascher[5]) to lose their flagella and become amoeboid. In the case here described, however, there is a transformation of a simple amoeba into a relatively complex flagellate which multiplies for several days then returns to the amoeboid condition and becomes encysted. Tetramitus rostratus has been studied by a number of investigators since its discovery in 1852, by Perty,[1] yet no one has given a full account of its life history. The lack of cysts in previous accounts, mentioned by Dobell and O'Connor,[6] is explained by the transformation into an amoeba before encystment. It is believed that this animal presents the extreme in transformations of this sort and serves to emphasize the close relationship between amoebae and flagellates, and the need for careful studies of life cycles, in pure cultures where possible, in investigations of coprozoic and other Protozoa. It is proposed to record at this time a brief account of the life history of this amoebo-flagellate reserving the more extended discussion for future publication.

The amoeba in question was first observed on November 5, 1920 in coecal material which had been taken from the rat and placed in sterile physiological salt solution on October 28, 1920. This is known as culture 10. After discovery of the amoeba, transplants were immediately made in various media. The amoebae were subcultured several times on solid media over a period of about two months, then two cultures were made in fluid media. In one the formula given by Sellards,[7] namely, 1.0 gr. Witte's peptone, 0.5 gr. lactose, 1000 cc. distilled water was used; in another, dextrose was substituted for lactose in Sellards' formula. The medium containing dextrose proved to be much more favorable for the amoeba under consideration. In these liquid media a flagellate appeared which is thought to be Tetramitus rostratus Perty.

In these cultures differences among the active amoebae as well as three types of cysts and the flagellate were noted, hence pure lines were started by the isolation method. Pure lines derived from culture 10 were designated G., I., J. and L.; of these G. and L. are of more special interest, the former being obtained from a single specimen of Tetramitus, the latter from a smooth, thin-walled cyst.

The cysts of cultures G. and L. revealed no microscopic differences either in the living or in the fixed and stained condition, and when transplants were made of the cysts in favorable media both active amoebae of limax type and Tetramitus developed. Pure lines started from an isolated amoeba, an isolated flagellate or an isolated cyst in liquid or semi-liquid media at room temperature have always given the same results, i.e., smooth, thin-walled cysts, the active amoeba of limax type and Tetramitus rostratus.

Although the amoeba with the Tetramitus stage was obtained in cultures from the coecal contents of the rat, the presence of contractile vacuoles etc., indicated that this amoeba was not a parasite but belonged to the coprozoic protozoa as defined by Dobell and O'Connor.[6] Feeding experiments upon rats and mice were conducted to determine whether the amoeba and flagellate stages are present either in the small intestine or in the coecum of these animals. Prior to feeding these experimental animals their faeces was collected on filter paper and cultures made in liquid media; in a few cases the amoeba and tetramitus stage was already present. The nine rats and four mice used were killed and opened in one, two, three, and five days after feeding and microscopic examination made of the fresh contents of the coecum and small intestine. Neither active amoebae nor Tetramitus were found, but cysts, both degenerate and normal, were present in all the animals. Cultures were made of coecal contents in sterile liquid media which were studied the following day. The active amoeba of the type used in the feeding and Tetramitus were found in these cultures. The results of the feeding experiments indicate that the amoeba under cultivation belongs to the coprozoic Protozoa.

Much evidence that Tetramitus is a stage in the life history of this coprozoic amoeba was obtained from the study of the living animal and from fixed and stained cover glass preparations, yet to affirm this fact with certainty, many consecutive hours of many days have been spent in close observation. Much of this study has been conducted on hanging drops, made either from a single cyst isolated and placed in a drop of fresh liquid medium or a few cysts taken by a platinum loop from a culture containing only cysts and adding from two to three loops of fresh medium. All were placed on depression slides and ringed with hard vaseline. By this method the animals have been kept under observation during the entire life cycle, a period of about four days. A brief description of the living animal during the stages of its life cycle will now be given.

1. *The Cyst:*—(fig. 1, A) *a.* Morphology. Commonly spherical, occasionally broadly oval, single, smooth, relatively thin wall; usually uninucleate; rarely binucleate; size ranging from 6μ to 18μ.

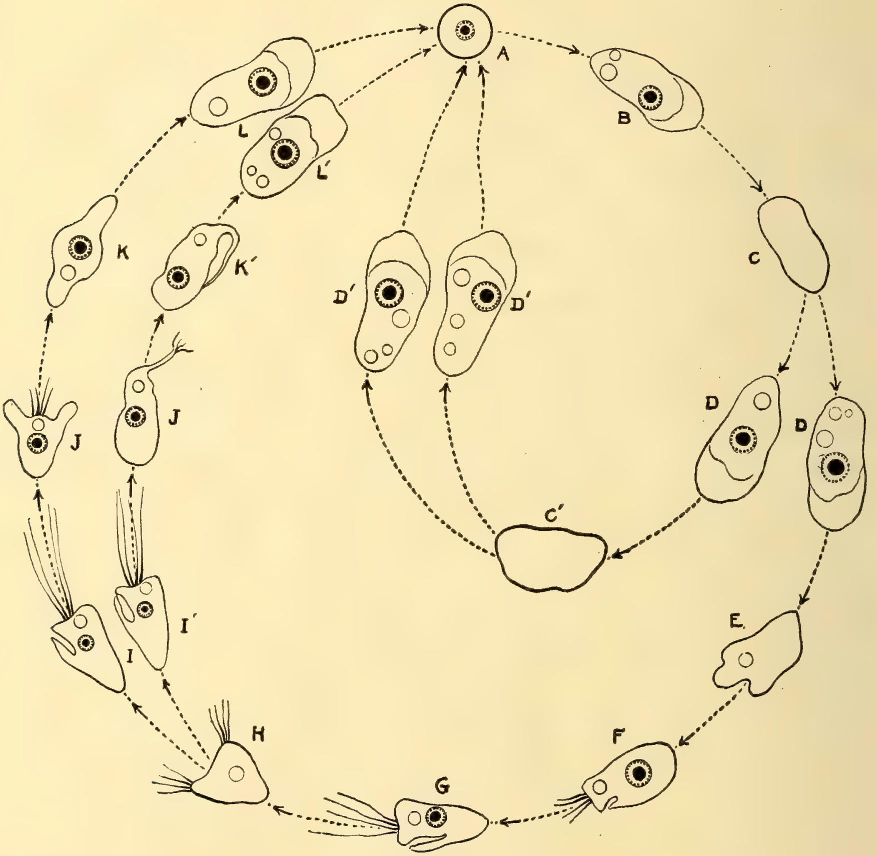

FIGURE 1

Diagram of life-cycle of coprozoic amoeba with Tetramitus as its flagellate phase. A—the cyst; B—vegetative amoeba; C, C'—division ("gel" state); D, D'—amoebae after division; E, F—phases in transformation of Tetramitus; G—fully formed Tetramitus; H—Tetramitus previous to division ("gel" state); I, I'—Tetramitus after division; J, K, L and J', K', L'—series of transformation stages, Tetramitus to amoeba.

b. Excystation. No empty cyst wall remains after the amoeba has emerged from the cyst, the wall apparently undergoing solution during the process of excystation.

2. *Amoeba:*—(fig. 1, B) *a.* Vegetative stage. Up to the present date the animal has always been observed to leave the cyst stage as an amoeba, never as a Tetramitus. The amoeba varies from 14μ to 48μ in length.

The rounded vesicular nucleus is usually 5μ to 6μ in diameter, and contains a caryosome 1μ to 2μ less in diameter; the pericaryosomal zone is usually clear but may contain a few granules. Peripheral chromatin on the nuclear wall in the form of granules can sometimes be observed in the living condition and generally in the fixed and stained specimens. The cytoplasm becomes sharply defined into a non-granular ectoplasm and granular endoplasm during the protrusion of pseudopodia. The amoeba generally moves by means of one broadly lobose pseudopodium, which however, may not be withdrawn before additional ones are formed. There is always one large contractile vacuole, augmented frequently by a few smaller ones.

b. Division. Gradually the rapid movement of the granules and of the nucleus through the endoplasm becomes slower, the pseudopodium is retracted, the granules clump together, then lose their identity so that the entire protoplasmic body becomes homogeneously refractive, and thus a "gel" state is formed during which the nucleus is obscured (fig. 1, C). The animal may remain quiescent in this state for from thirty seconds to several minutes, after which it suddenly elongates, constricts in the middle and divides into two daughter cells, meanwhile passing from the "gel" to the "sol" state (fig. 1, D, D′). The formation of a "gel" state during the process of division in higher animals has been noted in many papers by a number of observers, among whom are Chambers[8] and Heilbrunn.[9] Multiplication may continue for a number of generations without flagellation, or the amoeba may pass through a flagellate phase. Finally in each case the amoeba becomes encysted.

3. *Flagellate Phase:*—(fig. 1, G). Tetramitus varies much in size and in form as has been noted by Perty,[1] Fresenius,[10] Dallinger and Drysdale,[11] Stein,[12] and Klebs,[13] but the typical form is that described by them as "conical" with the pointed end directed posteriorly. The contractile vacuole and cytostome are located at the flattened anterior end which bears four flagella. The nucleus is slightly posterior to the contractile vacuole, and is similar in organization to that of the amoeba just described except that it is smaller measuring 3μ to 5μ in diameter. The animal is usually 14μ to 18μ in length and 7μ to 10μ in greatest width.

4. *Transformation of an Amoeba to Tetramitus:*—After the amoeba has passed through one or more generations by division, some of them may transform into flagellates. While the morphological changes may vary during the process, the results of one observation will now be given in detail. An amoeba retracted its pseudopodium, but changes in outline

continued slowly; the cytoplasm became homogeneously refractive and the nucleus was obscured probably due to a "gel" state (fig. 1, E). The contractile vacuole continued to pulsate slowly; an involution of the lateral margin near this organelle was formed which terminated in the protoplasmic body as a cleft. The animal now assumed an elongated somewhat pyriform shape, the margin near the vacuole being flattened. From this area flagellar activity began, and thus it became oriented as the anterior end. The nucleus, nearer the posterior end of the animal became demarcated from the homogeneously refractive cytoplasm (fig. 1, F). The animal began to move rapidly, undergoing many contortions, during which the cytoplasm became resolved into refractive granules suspended in a clear substance, the nucleus moved towards the anterior end and the pellicle became more firm so that the flagellate shape was assumed (fig. 1, G). The flagella had reached the average length and the characteristic rotation around the longitudinal axis began as the animal swam swiftly about.

6. *Division of the Flagellate:*—In a short time after the above described change the animal suddenly became broadly triangular in form (fig. 1, H), flagellar activity being observed at each angle limiting the anterior end. The animal then passed into the "gel" state and this was followed by longitudinal division (fig. 1, I, I′). The entire transformation, followed by fission in the above case, was completed in one hour, but these processes are not always so rapid. The flagellate may continue to multiply for several hours or even days commonly dividing into two by longitudinal fission but occasionally into three or four equal or unequal offspring.

7. *Transformation of Flagellate into an Amoeba:*—Eventually the Tetramitus transforms into an amoeba unless death prevents. The morphological changes which form this sequence are not always the same, but one set of observations will be given in detail. A Tetramitus suddenly shortened and became broadly triangular, two short processes were protruded from the angles of the flattened anterior end (fig. 1, J). Soon the body became rounded, the processes longer (fig. 1, K), and the movements slower resulting in but slight changes of location. Meanwhile the protoplasm exhibited external form changes marked by contortions of the body. The successive changes were very rapid and suggested that an effort was being made by the animal to free itself of flagella. Finally it became quiescent a few moments and then changed into the form of the typical amoeba which progressed by a pseudopodium of lobose type instead of by flagella (fig. I, L). Another series of changes are indicated by fig. 1, J′, K′ and L′. The transformation process is sometimes prolonged through many hours. The amoebae thus derived may multiply by division but ultimately they become encysted completing the life cycle.

The time periods occupied by the different stages in the life cycle at room temperature, as a rule, have been the following:—The cyst lives in a proper medium indefinitely. When cysts are planted on fresh media, amoebae may emerge in two or three hours and remain active from three to four days, rarely longer, and then encyst. The Tetramitus may appear in ten to twelve hours after transplantation of the cysts, continue numerous for two days and gradually disappear the third day as these become transformed into amoebae or die.

Exceptions to the above have occurred in two cultures in which activity of Tetramitus lasted for about a month. The study of the living animals together with the cytological examination of fixed and stained specimens from each culture revealed an increase in the number of nuclei to two, three and four, also the presence of giants and a greater proportion of animals of irregular shape.

While great numbers of these flagellates have been under observation for more than a year, no stages in the life history have been observed other than those described above. Binary fission for Tetramitus has been noted by Perty,[1] Dallinger and Drysdale,[11] Stein,[12] Klebs,[13] and Alexeieff.[14] Although no sexual stages nor spore formation such as described by Dallinger and Drysdale have been observed in these studies, nevertheless these observers did see the flagellate become partly amoeboid and a longer observation should have revealed to them the complete change to an amoeba.

In a life cycle of this kind, in the absence of sexual phenomena it is uncertain whether the amoeboid or the flagellate phase should be considered as the adult or dominant one, especially since non-sexual reproduction is extensive in both phases. A thorough study of the life cycles of other "amoebae" and other "flagellates" might reveal further examples of transformations of the kind here described.

In closing I wish to express my great indebtedness to Doctor D. H. Wenrich who has been ever ready to aid this investigation by participation and by fruitful suggestions. Also for the continued interest and aid of Professor Clarence E. McClung who has helped to surmount the many difficulties encountered and who, with the Graduate School of the University of Pennsylvania, has provided the facilities of the Zoölogical Laboratory for this research.

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A FURTHER NOTE ON THE AGE INDEX OF A POPULATION[1]

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In an earlier paper in these PROCEEDINGS, Pearl[2] proposed the function

$$\phi = S\left\{\frac{\Delta^2}{P}\right\}(M - M_p)$$

as a single numerical index of the age distribution of a population. In this expression S indicates summation, for all age groups given in the original statistics, of the expression in brackets; Δ is the deviation, for each age group, of the percentage of the actual population in each age group from the percentage of the same group in the standard population of Glovers[3] Life Table, denoted by P; M = mean age of living population; M_p = mean age of persons in a stationary population unaffected by migration.

In connection with certain problems now under investigation in this laboratory it becomes a matter of importance to know how sensitive this age index is to change in the size of the age groups of the original statistics. Since its original proposal the index has been used in many studies in this laboratory, and the larger experience has strengthened our confidence in its reliability as an index of significant variations in the age constitutions of populations. But it has always been used hitherto with at least 6 to 8 age groups covering the life span. Suppose the original statistics furnish only 3 age classes for the entire life span. Will this age index ϕ then give a reliable picture of the significant variations in age distribution, as we pass from city to city, or county to county?

To test this point, the obvious thing to do is to determine the correlation between the age indices for n communities, on the basis of say 3 divisions of the life span, with the age indices for the same communities on the basis of say 6 divisions of the life span. If the correlation is high it will mean that even with the coarse grouping the index is reflecting the main differences in the form of the age distributions.

The present note reports a test of this sort. The 1915 census of Iowa[4] gives in Table I the age distribution of the inhabitants of each of the 99 counties of the state in such form as to permit the formation of the following classes: 0–4, 5–9, 10–17, 18–20, 21–44, 45 and over, both figures being inclusive in each case. For each of the 99 counties ϕ was computed on the basis of this grouping of the ages. Call these values ϕ_1. Then for the same 99 counties the population was combined by addition into the 3 age groups 0–9, 10–20 and 21 and over. A new set of ϕ's was then calcu-

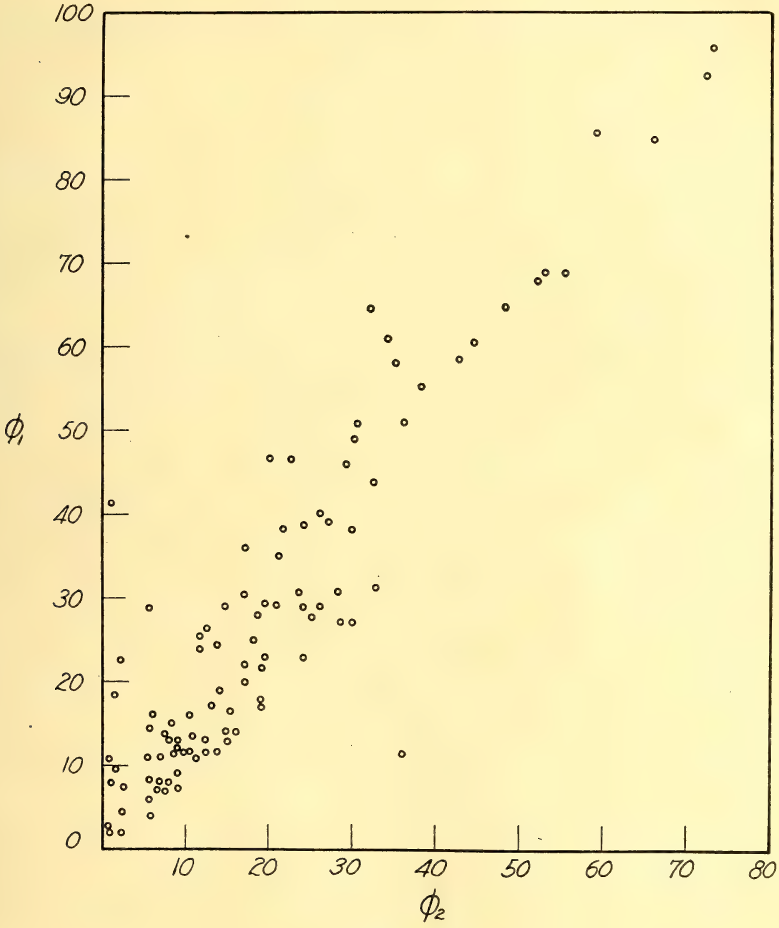

FIGURE 1

Spot diagram showing the correlation between ϕ_1 and ϕ_2. Each dot represents one county.

lated on the basis of this coarse grouping. Call these values ϕ_2. Then what we wish to know is the value of the correlation $r_{\phi_1\phi_2}$.

This was calculated by the direct product moment method, without grouping and gave as a result

$$r_{\phi_1\phi_2} = +.84 \pm .02.$$

It is at once evident that we are dealing here with a high degree of correlation. That the regression is linear is indicated by the spot diagram of the correlation surface shown in figure 1.

This result indicates that whenever it is impossible, by reason of unsatisfactory tabulation of the original raw statistics, to get any but a coarse age grouping, we may still use the age index ϕ with confidence that it will reflect closely the differences in population in respect of age distribution. The test here given is obviously a severe one, because all adult ages are thrown into one group. But even so, the correlation is high. With such a value of r as this, it is obvious that one could, by means of the appropriate regression equation, determine corrections for the ϕ_2 values, giving predicted or calculated ϕ_1's which would with extreme accuracy approximate the true values.

The means and standard deviations are as follows:

Mean $\phi_1 = 27.27 \pm 1.46$
Standard deviation $\phi_1 = 21.53 \pm 1.03$
Mean $\phi_2 = 19.71 \pm 1.23$
Standard deviation $\phi_2 = 18.14 \pm .87$

As would be expected the ϕ_2 mean is lower than the ϕ_1 mean, and ϕ_2 is less variable, as indicated by the standard deviation, than ϕ_1. The difference in the means is large, but this is of no great importance in the practical use of the age index. It is of much greater practical significance that the difference in the variabilities is not large. If we could trust here the theory of simple sampling (which we probably cannot) the difference between the standard deviations could not be regarded as certainly significant at all. The more important consideration is that ϕ_2 is reflecting in its standard deviation 84.2 per cent of the variation among these 99 Iowa counties in respect of age distribution of their populations, as measured by ϕ_1.

In conclusion it may be stated that since, as is demonstrated in this paper, there is a high correlation between the values of ϕ for extremely coarse age groupings, with the values obtained from finer groupings, this age index may be used with considerable confidence in cases where data on age distribution are available only for a few broad classes covering the life span. All our experience with this age index ϕ since its publication two years ago has confirmed our belief that it is a function of great value to the vital statistician who wishes to make analytical studies of medical and hygienic data by modern statistical methods.

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[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 69.

[2] Pearl, R. PROC. NAT. ACAD. SCI., **6**, pp. 427–431, 1920.

[3] Glover, J. W. *United States Life Tables, 1910.* Bureau of the Census, 1916.

[4] *Census of Iowa, 1915.* Published by the Executive Council.

THE THERMAL IONIZATION OF GASEOUS ELEMENTS AT HIGH TEMPERATURES

A CONFIRMATION OF THE SAHA THEORY

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Read before the Academy, April 25, 1922

In a recent article Eggert[1] has shown that the ordinary thermodynamic expression for the effect of temperature on the equilibrium of chemical reactions, when supplemented by assumptions as to the values of the special constants occurring in it, can be applied to the calculation of the extent to which under different conditions neutral atoms of the gaseous elements are converted into positive ions and free electrons. Saha[2] has shown that the most uncertain factor in Eggert's formulation can be eliminated by introducing the ionization-potential of the element. He has then calculated the dissociation of the first electron from the atoms of many important elements, thus, the extent to which such reactions as $Na = Na^{+} + E^{-}$ take place, where E^{-} represents electron gas. He has tabulated the values of this percentage ionization at various temperatures and pressures; and he has especially pointed out the great significance which these considerations may have for the interpretation of the spectra of elements under solar and stellar conditions.

These principles are of great interest to astrophysicists, as may be illustrated by the following applications. It has long been known that the solar prominences show strongly the hydrogen and helium lines and the enhanced (H and K) lines of calcium, but not the blue line of calcium nor the familiar lines of sodium or of other alkali elements. The remarkable absence of these lines had not been explained; but the thermodynamic relations now show that the extremely small pressure combined with the high temperature of the prominences must greatly promote the ionization of all elements, and may well suffice to convert the neutral calcium atoms into calcium ions, which produce the H and K lines; and also to convert completely the neutral sodium atoms, to which the D lines are due, into sodium ions, which do not yield strong lines in the visible spectrum. A second interesting application of Saha's equations has been made by Russell,[3] who was led to the discovery of rubidium in sun spots by the considerations that the absence of its lines in the general solar spectrum is probably due to the complete conversion by the high temperature there prevailing of the neutral rubidium atoms (which produce the characteristic lines of the element) into rubidium ions (Rb^+), and that at the lower temperature prevailing in sun spots this conversion might well be only partial.

The thermodynamic expressions may first be presented which are applicable to a reaction of the type under consideration, by which the neutral atoms of a gaseous elementary substance M are converted into positive ions and electrons in accordance with the chemical equation $M = M^+ + E^-$. At any definite temperature the equilibrium-constant K of such a reaction is expressed in terms of the partial pressures, p_M, p_{M^+}, p_{E^-}, of the three substances, regarded as perfect gases, by the following equations:

$$\frac{p_{M^+}\,p_{E^-}}{p_M} = K \qquad (1) \qquad\qquad \frac{x^2 p}{1 - x} = K. \qquad (2)$$

Unlike the first equation, which is general, the second equation, in which x represents the fraction ionized and p the sum of the pressures p_M and p_{M^+}, is valid only when $p_{M^+} = p_{E^-}$; that is, only when electrons do not originate from any other source, as from the presence of another ionizing element or from thermionic causes.

The second law of thermodynamics leads to a simple expression for the change of this ionization constant K with the temperature, in the case where the heat-content-increase ΔH attending the reaction can be expressed as a linear function of the absolute temperature T, thus by the formula $\Delta H = \Delta H_0 + T\Delta C_p$, where ΔH_0 is an empirical constant and ΔC_p is the increase in the heat-capacity at constant pressure that results from the occurrence of the reaction. The second law leads, namely, to the following equations, in which R is the gas-constant (1.985 calories per degree), and I is the integration-constant resulting from the integration of equation (3).

$$d \ln K = \frac{\Delta H_0 + T\Delta C_p}{RT^2}\,dT. \qquad (3)$$

$$\ln K = -\frac{\Delta H_0}{RT} + \frac{\Delta C_p}{R}\ln T + \frac{I}{R}. \qquad (4)$$

Now Saha introduces the following assumptions as to the quantities occurring in this equation:

(1) That the energy increase attending the ionization of a single molecule is equal to the electrical work Ve that must be expended in order to give to an electron a kinetic energy which just suffices to ionize an atom of the element with which it collides; e being the charge on the electron and V the ionization-potential, which last may be determined by direct measurement or computed from the spectral series of the element.

(2) That the heat-capacities of the neutral atom M and its ion M^+ are equal, and that the heat-capacity of electron gas is the same as that of any other monatomic gas, namely, 2.5 R per mol of the gas.

(3) That the constant I, which is the sum $I_{M^+} + I_{E^-} - I_M$, of constants characteristic of the three substances (the positive ion, the electron, and the neutral atom), can be evaluated by assuming $I_{M^+} = I_M$ and by calculating I_{E^-}, the constant for electron gas, by a theoretical principle that has been shown applicable to ordinary monatomic gases,[4] which need not be here described.

These considerations lead to the following values of the constants occurring in equation (4):

$$\frac{\Delta H_0}{R} = \frac{N\,e\,V}{R} = \frac{96500\ V}{8.316} = 11625\ V;\quad \frac{\Delta C}{R} = 2.5;\quad \frac{I}{R} = -15.10.$$

Substituting these values in equation (4) and changing from natural to ordinary logarithms, we get

$$\log K = -\frac{5048\ V}{T} + 2.5 \log T - 6.56. \qquad (5)$$

By this equation the ionization-constant K of an element at any absolute temperature T can evidently be calculated from its ionization-potential V (in volts); and from this ionization-constant its actual ionization can then be found by equation (1) or (2).

The main purpose now of this paper (of which a much more complete form will soon appear in the *Journal of the American Chemical Society* and the *Astrophysical Journal*) is to point out that the earlier investigation of Arrhenius,[5] and especially the more recent researches of H. A. Wilson,[6] on the electrical conductivity of flames into which salt solutions are sprayed clearly indicate that the conduction arises from an ionization process of the type here considered, and that these measurements make possible a computation of the relative magnitudes of the ionization-constants. Thus the experiments of these investigators have established the following facts.

1. The conductance of the flame is independent of the acidic constituent

of the salt; thus it is the same for the chloride and carbonate of any given alkali element. This is in accord with the assumption that these salts are completely decomposed in the way expressed by the chemical equations $2\ MCl = 2\ M + Cl_2$, or $M_2CO_3 = 2\ M + CO_2 + {}^1/_2\ O_2$, and that the conductance is due to ionization of the metallic element M which is common to its various salts.

2. The conductance changes with the concentration of any given salt and in mixtures of salts in the way required by equations (1) and (2), showing that the ions result from an ionization process in which one molecule dissociates into two others, as is the case in the reaction $M = M^+ + E^-$, where E^- denotes electron gas.

3. There is present in the flames both positive and negative carriers, but the mobility of the latter has been shown to be roughly 2500 times as great as that of the former, clearly indicating that the negative carriers are electrons and that substantially the whole conductance is due to them.

From the way in which the conductance changes with the concentration it was found possible by a mathematical process that need not be here described to calculate under the above stated assumptions relative values of the ionization constants K of the five alkali elements whose salts were investigated by H. A. Wilson. These relative values could then be compared with the absolute values calculated from the ionization-potentials by the thermodynamic equation (equation 5 given above). The results are reproduced in the following table.

TABLE 1

VALUES OF THE IONIZATION-CONSTANT ($10^{12}\ K$) AT 2000° ABSOLUTE

	Cs	Rb	K	Na	Li
From flame conductances	3500	780	430	5.6	0.8
From ionization-potentials	8200	1600	620	6.2	1.4

It will be seen that the ratios of the values for the successive elements run closely parallel to each other in the two series. Especially noteworthy is the very large change in both series in passing from sodium to potassium, and the much smaller differences between the other elements.

As stated above, the flame conductance data lead only to relative values of the ionization-constants; and the values given in the preceding table would contain an arbitrary multiplying factor if they were based on those data alone. It was found possible, however, by combining with the conductance data the results of previous rough determinations of the mobility of the electrons[7] and of the number of them per cubic centimeter[8] in entirely similar flames free from salt, to obtain absolute values of the ionization constants; and these are the numbers that are recorded in the first row of the above table. Although they afford an estimate only of the order of magnitude of the constants, yet they are seen to correspond well with the values calculated by the thermodynamic equation.

The results here presented therefore afford a striking confirmation of the general validity of the calculation of the ionization of elements from their ionization-potentials by the thermodynamic equation employed by Saha.

[1] Eggert, *Physik. Zs.*, **20,** 570, 1919.

[2] Saha, *Phil. Mag.*, **40,** 478, 809, 1909; **41,** 267, 1921.

[3] Russell, *Astrophys. J.*, **55,** 129, 1922.

[4] Sackur, *Ann. Physik.*, **36,** 598, 1911; **40,** 67, 1913; Tetrode, *Ibid.*, **38,** 434; **39,** 255, 1912; Stern, *Zs. Electrochem.*, **25,** 66–80, 1919; Tolman, *J. Amer. Chem. Soc.*, **42,** 1185, 1920; **43,** 1593, 1921; Lewis, *Physic Rev.*, **18,** 121, 1921.

[5] Arrhenius, *Wied. Ann. Phys.*, **43,** 18, 1891.

[6] H. A. Wilson, *Phil. Trans. Royal Soc.*, **A216,** 63–90, 1915.

[7] H. A. Wilson, *Phil. Mag.*, (6) **10,** 475, 1905.

[8] Wilson and Gold, *Phil. Mag.*, (6) **11,** 484, 1906.

AN ELECTRON THEORY OF ELECTRIC CONDUCTION IN METALS

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Communicated, September 5, 1922

This is a modification and extension of the dual conception of conduction which I set forth in a paper[1] printed several years ago and to which I have referred in more recent publications. The theory in its present shape, like every other theory of metallic conduction with which I am acquainted, involves propositions that may well be questioned, but it contains certain features not present, so far as I know, in other theories, and I hope these will be found worthy of consideration, even if they are finally to be rejected.

I shall endeavor to state this theory, without extended argument, in the form of certain propositions, some of which are or involve outright assumptions. Any helpful hypothesis, not in conflict with well established facts or principles, I have considered justifiable, in view of the difficulties of the problem with which I am dealing.

Behavior of the Ions in Conduction

1. Complete metal atoms are of such hardness and symmetry that they are not subject to appreciable polarization or orientation in impressed electric fields of ordinary strength.

2. An ion is formed from an atom by loss of an electron from the outer shell and is, accordingly, unsymmetrical, essentially polarized, and therefor subject to orientation in a uniform electric field such as may be ap-

plied to a metal from without. This proposition seems to be in general accord with Langmuir's conception of atomic shells. See no. 991 in *Science Abstracts* for 1919.

3. The behavior of these ions in an electric field is somewhat like that of magnetic particles in a magnetic field, as set forth in Langevin's theory[2] of paramagnetism, or that of electric dipoles as described by Sir J. J. Thomson.[3] But in one vital particular, indicated in the next paragraph, the formula used by Langevin and by Thomson must be modified to suit the present case.

4. When and where the equipartition law of energy holds, the Boltzmann-Maxwell distribution law can be written with the factor $e^{-V/RT}$, where V is the mean potential energy of a particle with respect to any coördinate, either of position or of direction, in the field of force, and RT is twice the mean kinetic energy of the particles with respect to any degree of freedom whatever. But when and where the equipartition law fails, we must, if we can use the Boltzmann-Maxwell formula at all, use it with regard to the particular coördinate or degree of freedom that we are dealing with. Kinetic energy along one coördinate may in this case have nothing to do with the distribution of particles along another coördinate. For example, if all the kinetic energy of air particles became rotary energy, the atomosphere would fall. So, too, if we are dealing with particles that have no rotary energy, the translatory energy of these particles will have nothing to do with their orientation in a directing field of force.

5. The atoms and the ions have a mean kinetic energy of rotation proportional to the absolute temperature but of a much smaller order of magnitude than that of the energy of translatory vibration.[4] Accordingly we should, in the Boltzmann-Maxwell formula, use a very much smaller value than Langevin and Thomson have used in the denominator of the exponent of e, thus getting a much greater orientation, in a given field of force, than their formula would lead us to expect. This orientation will, through a great range of electric field strength, be proportional to the strength of the field and inversely proportional to the absolute temperature.

6. The time required for the orientation of an ion to reach its final value is large compared with the period of the translatory vibrations of the ion. See (21).

7. An ion has on its surface a fairly definite scar or *pit* from which an electron has been dislodged, and the ion can capture an approaching electron, thereby recovering the status of an atom, only when the electron strikes, not too violently, within this pit.

8. A free electron, approaching an ion with the kinetic energy of a gas molecule, may, though attracted toward the pit, shoot by, failing to make

connection, as a comet shoots by the sun. Accordingly, termination of the life of an ion by the taking up of a free electron may be a comparatively rare occurrence, most of the ions ending by taking up electrons directly from neighboring atoms.

9. Though the atoms have a regular arrangement in a continuous metal crystal, the individual crystals in any ordinary piece of metal are so small and so heterogeneously placed with respect to each other, that we may think of the atoms and ions as, on the whole, subject to mutual collision with any direction whatever of their line of centres. Accordingly we may think of the *pits* as *forming* anywhere on the ions, in one quarter as probably as in any other, when no impressed electric field is in operation.

10. Impressed electric fields of ordinary strength have no appreciable influence in determining or preventing the passage of an electron from an atom to a neighboring ion.[5] Thus a potential gradient of 1 volt per centimeter, imposed from without, would be a very steep one, whereas, if the "ionizing potential" of an atom within the solid metal is of the order of 1 volt, as I believe it to be, the field of force which a neighboring ion may exert on an electron is of the order of 10^8 volts per centimeter. It is possible, however, that when extremely great current densities are used, as in attempts to pass beyond the range of Ohm's law, the impressed field may be great enough to have a perceptible effect in determining whether an electron shall stay on its present atom or pass over to an adjacent ion. Such an effect would tend to make the current increase more rapidly than the potential gradient and might do something to mask the opposite kind of change that would naturally come from the approach of the ions to a saturated state of orientation.

11. The impressed electric field does its work, under ordinary conditions, by turning the ions so that the pits upon them, wherever they at first occur, move as a positive charge would move, in the conventional direction of the field force, while the electron portion of the damaged surface shell of the ion moves as a whole in the opposite direction. Thus we have an electric *current*. The effect of the to and fro heat translations of the ion, superposed on the process of orientation, seems to be, practically, to enlarge the ion.

12. The rapidity of the process of orientation of an ion is probably controlled by various factors, one of which is, of course, the moment of inertia of the ion, but the time occupied in this process is assumed to be less than the normal life time of an ion. See (21).

Ohm's Law

13. Let the condition of the ion i in figure 1 represent the completed orientation due to a field the lines of which run in the direction of F. If the atom a, now in contact with i, loses an electron to i, a pit will form

on a, and the transfer will have substituted for an ion of negative potential energy an ion of equal but opposite orientation, that is, an ion of positive potential energy of orientation, and the new ion will at once begin to swing in such a way as to resume the process that had come to an end on the first ion. Other things being equal, the strength of the current will be proportional to the sine of the angle of orientation, α in figure 1, which through a wide range of field strength is (see Prop. (5)) practically proportional to this strength. Thus Ohm's law is satisfied, so far as the kind of conduction now under consideration, that is, the "associated-electron conductivity," which in my previous papers I have called k_a. The "free electron conductivity," k_f, is to be considered briefly later in this paper.

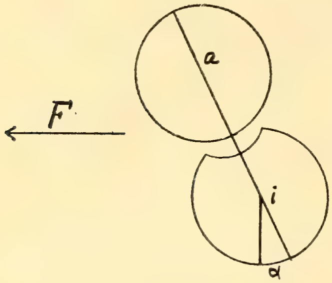

FIGURE 1

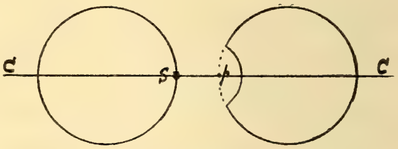

FIGURE 2

The Temperature Relation

14. It is misleading to speak of the conductivity of pure metals as being proportional to the reciprocal of the absolute temperature. If this relation held, the temperature coefficient of resistance would be 0.00367 for all metals, whereas its mean value between 0° C. and 100° C. is greater than 0.00367 in most metals and is greater than 0.006 in some. Moreover, the purer the metal, the higher the value of this coefficient. Accordingly a theory of conduction that makes conductivity inversely proportional to T proves too much. It is probable that the observed rather loose relationship between conductivity and temperature is the outcome of opposing influences, the direct effect of rise of temperature being to increase resistance much more rapidly than the temperature increases, while increase in the number of ions with heating of the metal tends to reduce the resistance. This idea is elaborated in the following paragraphs.

15. The necessary relative position of an ion and an atom, with a view to the passage of an electron from the atom to the ion, is indicated by figure 2, where the circle represents an atom, the imperfect circle an ion, p the point at which an electron must lodge in order to make an atom of the

ion, *cc* the line of centres, and *s* a little three dimensional space within which the point *p* must be in order to capture an electron from the atom. It is to be noted that the pit in *i* allows the point *p* to pass *through s* during a very close approach of the ion and the atom.[6]

16. Let us for convenience suppose the atom to remain at rest, all the relative motion of the two particles being, for the present discussion, attributed to the ion. Ignoring any small long-period rotary oscillations the ion may have, we see that its vibrations may be resolved along three axes, one the line *cc*, and the other two at right angles with *cc* and with each other. The little vital space *s*, within which the point *p* of the ion must be in order to take up an electron from the atom, lies at the middle of the second and third of these component vibrations but nearer one end than the other of the vibration parallel to *cc*.

17. The probability that *p* will gain an electron during any single passage through *s* is proportional to the length of time spent in this passage. It is, then, so far as the second and third axes are concerned, inversely proportional to the product of the amplitudes of vibration parallel to these axes, each of which amplitudes is proportional to $T^{0.5}$. As to the vibrations parallel to *cc*, the space *s* lies relatively nearer the end in a short vibration than in a long one. Accordingly it may be assumed, provisionally, that, so far as motion along *cc* is concerned, the time spent by *p* in passing through *s* will be inversely proportional to some power of T higher than $T^{0.5}$. The result, all three component vibrations being considered, is a probability, that *p* will gain an electron in a single passage through *s*, proportional to T^{-x}, where x is some quantity a little greater than 1.5.

18. The conductivity under discussion should, other things being equal, be proportional to the amount of orientation suffered by the ions, which according to (5) varies as T^{-1} and to the frequency of transmission contacts between ions and atoms, which according to (17) varies as T^{-x}. Hence if n, the number of ions per cu. cm., remained constant, we should have this conductivity proportional to $T^{-(1+x)}$ where $(^{1+x})$ is somewhat greater than 2.5.

19. In the series of papers on thermo-electric action, thermal conduction, etc., which I have published in these PROCEEDINGS[7] during the past few years, I have been led to the conclusion that the value of n can be usefully represented by the formula $n = zT^q$, where z and q are constants, the value of the latter varying in the many metals considered, from 1.2 in bismuth and in iron to 1.6 in nickel, palladium and platinum, the mean being 1.48. This general rate of increase of n with rise of temperature, taken with the propositions of (18), suggests that the conductivity k_a, now under discussion, should vary, approximately, inversely as the absolute temperature, though individual metals might depart rather widely from this relation. The relation in question is:

$$k_a \propto T^{-(1+x-q)}$$

20. Evidently it is possible, by taking for each metal an arbitrary value of x, to make the ratio of k_a at 0° C. to k_a at 100° C. agree with the observed ratio of total conductivity at 0° C. to total conductivity at 100° C., but this would be a rather fantastic performance. Attempts to account exactly, at present, by means of the theory of conduction presented in this paper, for the details of observed fact in the relationship of conductivity to temperature would be the opposite of convincing. The value 1.6 for x serves very well for many metals.

21. I cannot in this paper reproduce the discussion by which I have sought to test the quantitative adequacy of the theory here presented. This discussion involves necessarily considerable additional conjecture and is not yet to be regarded as conclusive, but it is distinctly encouraging, provided it is allowable to use for certain factors values of an order of magnitude indicated below:

Moment of inertia of an atom, 10^{-42} gm., cm.,^{2}

Time of orientation process, 5000 times the period of "characteristic" heat vibrations.

Life-time of an ion, at o° C., twice the orientation time.

n, the number of free electrons or of ions, per cu. cm., in general accord with the data given in my *Summary,*[7] though I am now inclined to take, at 0° C., the mean path of a "free electron" within a metal as perhaps fifty times the centre to centre distance of the atoms, instead of ten times, and so get a correspondingly smaller number for n, one of the quantities varying inversely as the other.

Under these conditions the mean kinetic energy of rotary motion of an atom (see (5)) comes out about 10^{-9} times the energy of translatory vibration.

Behavior of the Free Electrons in Conduction

22. According to the data given in any *Summary,* free-electron conductivity bears to total conductivity about the same ratio at 100° C. as at 0° C., that is, in most metals the free-electron conductivity, k_f, is roughly inversely proportional to the absolute temperature.

23. Various writers give us the formula

$$k_f = \frac{n \lambda c e^2}{\kappa R T},$$

though they may differ in the value assigned to the constant κ. If we take as approximately true, the proposition that $k_f T$ is a constant, we have

$n\lambda c$ equally constant, and so, if we take $n \propto T^{1.5}$ (see (19)), we have $\lambda \propto T^{-2}$, approximately.

24. In the attempt, not hopeless, which I have been making of late to explain the various "transverse" effects in accordance with the propositions of my *Summary*, I have been led to the conclusion that the diameter of the outer electron shell of an atom may be as small as one-half the means centre to centre distance of neighboring atoms. This conception makes it comparatively easy to accept the proposition that λ may be, as already indicated in (21), fifty[8] times as great as this centre to centre distance.

If λ really extends through many one-atom-thick layers of the metal, a slight decrease in the probability of an electron's passing through any single layer would be enough to account for the rapid decrease in λ with use of temperature, as found in (23).

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[1] "A Possible Function of the Ions in the Electric Conduction of Metals," Proc. Nat. Acad. Sci., Vol. 3, March 1917.

[2] *Journal de Physique*, series 4, vol. 4, pp. 690–692.

[3] See pp. 86–88 of his *Molecular Constitution of Matter*.

[4] Borelius makes a like assumption regarding the kinetic energy of what he calls free electrons. See *Ann. Phys. Chem.*, vol. 57, p. 233.

[5] In the paper referred to in footnote (1), I held quite the opposite opinion, making the effect of the field at this juncture the basis of my explanation of Ohm's law.

[6] If this conception is sound, it would not be surprising to find very great pressure unfavorable to the state of superconductivity.

[7] For a *Summary* see Proc. Nat. Acad. Sci., **7,** No. 3, March 1921.

[8] Bridgman has of late taken λ to be "many" times the atomic diameter.

STATIC DEFLECTION, LOGARITHMIC DECREMENT AND FIRST SEMI-PERIOD OF THE VACUUM GRAVITATION NEEDLE*

By Carl Barus

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Communicated, August 12, 1922

1. *Apparatus.*—The object of the present experiments is a completion of the work of an earlier paper,[1] by carrying the exhaustion of the case as far as practicable.

In spite of the care taken to seal all parts of the extended apparatus, there remained a constant leak of .0035 mm. of mercury per hour, the seat of which I was unable to detect; but as experiments with gradually decreasing vacua were primarily contemplated, this leak was here no serious disadvantage, except that the highest exhaustions were not available. It is of course possible to keep the pump running and the stopcock open. When this is done, however, the needle (quite apart from tremor) is always in motion, so that gravitation measurements are out of the question.

2. *Data. Logarithmic Decrement,* λ.—In the work of last year the logarithmic decrement as obtained from observations largely made at night, seemed to remain fairly constant until the highest exhaustions were approached. This result needed the further qualification undertaken in the present paper. To compute $\lambda \log e$, the two arcs obtained from the first three elongations of the needle $y - y'$, $y' - y''$ were used. In view of the specified small leak in the apparatus, the data for λ could be found with great accuracy in this way in the course of time, while the vacuum slowly degenerated from about .0002 to .271.

These data and the vacua to which they belong (observations being made in the morning (A), afternoon (P), and night (N)), are constructed in relation to the vacua (mm. of mercury) in figure 1. F denotes that the attracting weight M, on the right, is in front, and R to the rear of the case and needle, the opposite conditions holding for the M near the left end of the needle. The first arc of swing was usually between $y = 20$ and $y = 30$ cm., depending on the time of day.

Inspection of figure 1 shows that changes of viscosity due to changes of pressure, are not discernible in results of the present kind. In fact the logarithmic decrement at the highest exhaustions (July 5) happens to be larger than at the lowest exhaustions (July 8, N). The logarithmic decrement is therefore, also, primarily under the influence of the radiant forces. It is largest in the morning, and least at night. It obviously oscillates once per day, though I did not make observations after 11 P.M. If $d\theta/dt$ be the change of atmospheric temperature per second in the environment of the apparatus, then this coefficient is the controlling factor in the marked variations of $\lambda \log e$.

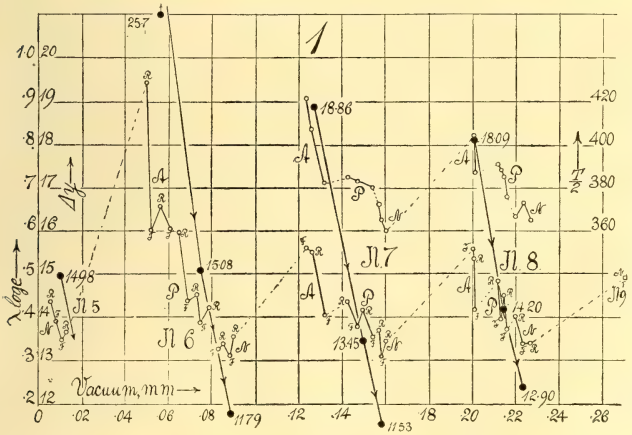
Moreover the values of $\lambda \log e$ are always larger for the rear positions R of M, than for the front positions F, under otherwise like conditions. One or the other, or their mean, must therefore be taken in association with $d\theta/dt$. This may be due to a lack of symmetry of the position of the needle to the case; but it is more probably due to a lack of symmetry in the environment.

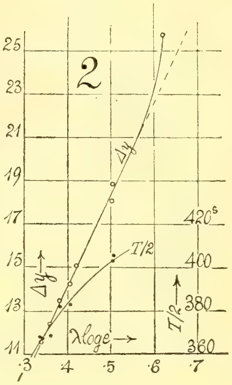
3. *First Semi-Periods.*—The accurate method of finding $T/2$ from two successive passages of the needle through the position of equilibrium is very troublesome; for $T/2$ is long and the position of equilibrium varies. Similarly the variations of $T/2$ are large, and precise values of it are here apparently of little use. Hence the measurement of $T/2$ from elongation to elongation was accepted as adequate. These values, so far as taken,

are constructed in figure 1, on the right, in seconds. The variations may run from 6 to 7 minutes and the $T/2$ values are largest in the morning and least at night. They march about in parallel to the λ values, if either the R or the F position alone is taken. T thus also varies harmonically in the lapse of time with a period of 24 hours referable to $d\Theta/dt$, provided the first semi-period, $T/2$ is taken.

From these results it might be surmised that the true period T can be computed from the log decrement observed at the same time. But it is easily seen that the correction so obtained is of an order entirely too small to account for the existing divergencies. The theory of the damped pendulum is only useful here as an analogy.

4. *Static Displacements*, Δy.—Continuous series of these data were given in the earlier papers and obtained from night observations (7–11 P.M.), when they are liable to be most constant. The black dots in figure 1 are the present values for different parts of the same day. Δy is the





static displacement finally obtained, when the attracting weights M are swung from one side of the needle to the other. As figure 1 shows, Δy is now also markedly harmonic in the lapse of time, or of the reduced exhaustion proportional to time, with a period of one day.

We thus come to the result that λ, $T/2$, Δy are similar time functions; that as a first approximation they are independent of the change of viscosity of rarified air and that their relative variation is such as to admit of their expression in terms of each other. It does not necessarily follow that a correct value computed for one (for instance for λ) would lead to correct values for $T/2$ and Δy; but it is a project well worth testing.

5. *Comparison of* λ, $T/2$, Δy.—It is next necessary, therefore, to more specifically compare all the quantities obtained in the present investigation. This has been done in figure 2, where the abscissas are $\lambda \log e$ and the ordinates $T/2$ and Δy, respectively.

Turning first to the latter, it is astonishing to find that λ and Δy, in spite of the marked amplitude of radiant forces, make up a definite graph. In fact, if we omit the exceptional $\Delta y = 25.7$ (inadmissably large radiant forces), Δy and λ are nearly proportional. If this curve were known, Δy could be computed from λ and vice versa.

The values of $T/2$, as already intimated, were not accurately taken, as I did not anticipate relations like the present. Nevertheless figure 2 adequately shows that these also make out a definite graph.

If for instance $\Delta y = 13.4$ cm.(about the mean value in the preceding paper), $T_1 = 758$ sec. should be its equivalent and $\lambda \log e = .385$ at the same time. If we take the vacuum period as $T_1/\sqrt{1+\lambda^2/4\pi^2}$ it would be $T = 750$ sec. which happens to agree with the value taken in the report in question.

6. *Plenum.*—In contrast with the preceding oscillations under conditions of high exhaustion, the results investigated for the case of the needle vibrating in air under atmospheric pressure, are nearly aperiodic. The data for T, λ, Δy, are all enormously larger. These values of λ result from the continued drift of the position of equilibrium of the needle toward the direction in which it is deflected by the attracting weights M. In this respect the plenum results are quite different from the preceding under exhaustion.

The motion of the needle in a plenum is peculiar. When the weights are exchanged, the needle falls from its high elongation to its low elongation in the time $T/2$. It then turns toward a new high elongation for a time (usually about $T/4$ sec. in length), after which it again moves toward low numbers by slow indefinite creeping, without again turning. Exactly the same phenomenon occurs when the needle rises from a low elongation toward a higher. In other words the needle may be said to oscillate about a position of equilibrium continually advancing in the direction of the deflection.

7. *Comparison with Theoretical Values of Frictional Resistance.*—It is now desirable to endeavor to construe the values obtained for λ and T by the aid of the familiar theory of the damped pendulum. If the needle be regarded as consisting of the two balls at its end, the frictional forces should be $6\pi\eta r v$ and therefore the frictional coefficient b is $6\pi\eta r$. If T_1 is the period of the damped needle and λ its logarithmic decrement $T_1 b = 2\lambda$, and therefore, $\lambda = 3\pi\eta r T_1$. Hence, if $r = .23$ cm., $\eta = .00019$, $T_1 = 892\lambda \log e$ seconds. In figure 2, however, not only is proportionality of T and λ at long range excluded, but the factor would be twice as large, on the average.

---

* Advance note, from a Report to the Carnegie Institution of Washington, D. C.

[1] These PROCEEDINGS, **8,** pp. 13, 63, 1922. The present apparatus is essentially the same.

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PROCEEDINGS
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Volume 8 NOVEMBER 15, 1922 Number 11

BACTERICIDAL PROPERTIES OF THE PRODUCTS OF RADIUM EMANATION

By John Albert Marshall

College of Dentistry, University of California

Communicated, October 7, 1922

The following abstract deals with a description of the use of radium therapy in periapical infections,[1] and supplements a preliminary note[2] in which it was pointed out that solutions of the products of radium emanation possess distinct bactericidal properties. Additional data have been compiled but the analysis of them clearly indicates that with the present-day knowledge it is impossible to state definitely and conclusively why the products of the emanation produce such characteristic effects.

Although it was reported in 1909[3] that radium exerts an inhibitory influence on pathogenic and non-pathogenic organisms, no further work was done in this field until ten years later, when the former observations were confirmed.[4] Bacteria of various types were exposed to radium emanation with the result that a distinct inhibitory effect upon the cultures was observed.

In this present experiment a *solution* of the products of radium emanation is employed. The emanation tubes are usually discarded by the laboratories after they have undergone disintegration in respect to their content of gamma ray. However, they still contain in varying proportions radium A, B, C, and D. The effects which have been obtained are presumably not due to the hard gamma ray, but result in all probability from the action of the softer beta ray.

The radioactive solution is prepared as follows: a "de-emanated" emanation tube is placed in a sterilized mortar of about 20-cc. capacity, covered with Ringer's solution and then crushed with a pestle. On account of the rapidity with which the radio-activity of this solution diminishes, it is advisable to crush the tubes immediately before use. This solution is carried directly to the infected area by means of sterile dressings. In the case of chronic or acute apical abscesses, found about the roots of teeth, the radioactive liquid is forced into the infected area through the pulp canal of the tooth and allowed to remain for 24 hours. Subsequently the dressings are removed, the canals tested for sterility and finally filled. With two possible exceptions there has been no evidence, after three years time, of succeeding soreness or pain. The individual dosage in terms of millicuries of gamma radiation has varied from .046 to .138.

Failures in this method of treatment have been recorded but the fact remains that sterile tests have been obtained more often by using the radioactive solution than in similar experiments with other antiseptic agents.

Summaries of experimental and clinical data are presented which indicate the variations in the pathology, age of patient, etiology, treatment, clinical result and laboratory findings.

In many instances tubes have been used in which de-emanation had proceeded to such a degree that it was not possible to obtain a reading by the electroscope. These have given similar results insofar as bactericidal action is concerned, but they have not caused the soreness which sometimes accompanies the use of solutions containing a recorded amount of gamma rays.

In general the cases selected have been those in which previous treatments were without satisfactory result. Dichloramin T, formocresol beechwood cresote, phenol, and other antiseptics of a similar nature had been used without producing a noticeable change in condition. In fact, the greater majority of the cases carried a history of unsuccessful treatment. Most of them yielded to the action of the radioactive solution.

The precautionary measures which are to be observed are naturally those referring to dosage and technique of application. Severe radium burns resulted in monkeys when emanation tubes containing eighteen millicuries were sealed into pulp canals of central incisors. These burns were so extreme that the hard palate and orbit were involved. It is believed to be impracticable to use the radium except as a solution of the solids. The tubes should be thoroughly crushed under Ringer's solution exactly at the moment of use, and it has been customary to have three or four patients prepared and ready at one time in order to secure the greatest economy of material. The limited availability of de-emanated tubes except at emanation laboratories is a serious drawback to an extensive application of this therapy. However, it is planned, if funds become available, to further extend these researches to include not only additional data upon dosage but also to report upon the application of other radioactive preparations which may be found to be more plentifully distributed.

[1] Marshall, John A., "Radium in Periapical Infections" (*in press*).

[2] Marshall, John A., "Bactericidal Action of Solutions of Radium Emanation," *Calif. State Med. J.*, Jan. 1922.

[3] Iridell and Minett, "Notes of the Effect of Radium in Relation to Some Pathogenic and Non-Pathogenic Organisms," *Lancet,* 1909, **1**, p. 1445.

[4] Lequeux and Chrome, "Action of Radium on Bacteria," *Arch. Memo. Obst. gynec.*, **3**, p. 698, Dec. 1919.

THE JADES OF MIDDLE AMERICA[1]

By HENRY S. WASHINGTON

GEOPHYSICAL LABORATORY, CARNEGIE INSTITUTION OF WASHINGTON

Communicated October 4, 1922

The following pages present some of the results of a study of jade beads, disks, and other small objects, found in a *cenote,* or sacred natural well, at the ancient Maya city of Chichen Itza, in northern Yucatan, the petrographic study of which was kindly entrusted to me by Prof. A. M. Tozzer of Harvard University. The final results will be incorporated in a forthcoming archaeological monograph on the contents of the cenote, under the editorship of Prof. Tozzer. In connection with this study, I have also been able to examine some ancient Mexican beads, through the kindness of Dr. L. Salazar, Director of the Geological Survey of Mexico, and some jade objects from Copan, Honduras, which were kindly placed at my disposal by Dr. Sylvanus G. Morley. I am deeply indebted to Dr. H. E. Merwin, of this Laboratory, for numerous optical determinations.

There is great variety in all the characters, chemical, mineral, and physical, of the Chichen Itza and other American jades. There are two dominant colors: green and gray. Some objects are wholly of a deep, rather yellowish green, like Ridgway's "pistachio green," or of a grayish green, like Ridgway's "pea green," the first being used for the choicest objects. The grays vary from almost pure white, through ash-, gull-, and dove-gray, to rather dark gray, most of the gray specimens being mottled in lighter and darker shades. Another and common type is mottled white (or very light gray) and a clear, bright, grass-, or emerald-green. A few obsidian beads were also found at Chichen Itza.[2]

The texture varies from decidedly coarse-grained, as are most of the pistachio-green and mottled green and white specimens, to very fine grained, as are most of the pea-green specimens; the gray specimens are mostly moderately fine-grained. The coarse-grained pieces show glistening cleavage surfaces of pyroxene on the fracture. Nearly all these objects have a high to very high polish, which is most marked on the coarser and the pistachio-green varieties. The coarse varieties are rather translucent, as are some of the fine-grained pea-green beads; the whites and grays are generally opaque.

A few beads are dead white, of dull luster, and slightly crumbly. These have evidently been subjected to the action of fire, and the effect of this is clearly seen in one or two long beads which have been heated to the point of softening, so that they are bent and warped to such an extent that the longitudinal perforation is no longer straight.

The coarse-grained, translucent, green or green and gray varieties have a hardness of about $6^3/_4$, while the mottled gray and white kinds are about $6^1/_2$. None of this American jade shows the extreme toughness which is so characteristic of most jade, especially nephrite.

Many density determinations have been made, representing the various color and textural varieties, from Chichen Itza, Copan, and Mexico. The density varies from 3.335, about that of pure jadeite, to 2.667, about that of oligoclase. Some characteristic densities are given later. The numerous determinations by Hallock, published by Clarke and Merrill,[3] also show the wide variation in density.

In thin section most of the coarser specimens, whether wholly green, mottled green and white, or wholly white or gray, are seen to be composed entirely of anhedral or subhedral grains of pyroxene, no other mineral being present. The pyroxene grains are somewhat cracked, but there is little other evidence of crushing. The megascopically white and gray material is perfectly clear and colorless, while the green specimens, and the green portions of those which are mottled, are of a light, slightly yellowish green, with slight pleochroism. The green color is obviously not caused by the aegirite molecule. In the green-white mottled specimens, the color is distributed like a stain, covering small irregular areas, and fading out gradually in all directions, so that some grains near the border are partly green and partly colorless. In some pieces the green color is seen megascopically to be deeper along fine cracks, indicating the presence of a chromium-bearing solution. The purely pyroxenic specimens, it may be said, are those of the higher densities.

In other specimens, of the lower densities, and chiefly among the mottled green and white, mottled white and gray, and the fine-grained light pea-green varieties, there is more or less feldspar, which appears as a water-clear albite, in irregular interstitial areas or a granular mass. This albite very rarely shows twinning. The relative amount of albite (or oligoclase) varies within wide limits, from a few shreds and patches to the greater part of the section. As the amount of albite increases the pyroxene grains become smaller, more prismatic, and more scattered, until in the most albitic specimens the small percentage of pyroxene forms small prisms scattered through a matrix of albite grains.

Apart from the pyroxene and albite, no other minerals are present, except for a very few small shreds of muscovite and an occasional very minute magnetite grain.

The pyroxene is of very variable composition, but is composed essentially of jadeite and diopside, the proportions of which vary within wide limits, from jadeite : diopside = 21 : 1 in No. 6, to about equal amounts in the Tuxtla statuette,[3] and even 1 : $2^1/_2$ in the Copan pebble.[4] Jadeite can evidently take up considerable amounts of diopside in solid solution. The material of the Tuxtla statuette, with about equal amounts of each, may be regarded as an end member of the series, and for this the name *tuxtlite* (pronounced tushtlite) is proposed. It may be a double salt, analogous to diopside or dolomite. Its optical characters (as determined by Merwin) are given in the paper cited, as well as its density (determined by Adams) and its chemical composition. From the data given later in table I, it would appear that the diopside-jadeite can take up albite and anorthite in solid solution to a certain extent, although according to Bowen,[5] there is no experimental evidence of the solution of plagioclase in typical diopside, $CaMgSi_2O_6$.

Some of the chemical analyses made by me of various Middle American jades, with two made by other analysts, are presented in table I. The densities and the optical properties of the pyroxene were determined on the same fragment which was used for the analysis. For the optical determinations I am deeply indebted to Dr. H. E. Merwin. With each analysis is given the composition calculated in terms of the four molecules: jadeite, diopside (including in some cases very small amounts of hypersthene, or wollastonite and a little excess ferric oxide), albite (including minute amounts of orthoclase), and anorthite. After allotting alumina to potash and soda, the invariable small excess was allotted to lime in anorthite. Diopside was then calculated, the usual slight excess of magnesia and ferrous oxide over lime forming very small amounts of hypersthene. The residual silica was, in every case, more than enough to form jadeite and less than enough to form albite; so it was distributed between these two molecules by the usual equations used in calculating the norm.

The first five specimens are shown by the microscope to be quite monomineralic, composed entirely of pyroxene, and without any feldspar, although No. 5 shows a very few small shreds of muscovite. The section of No. 6 shows a little albite, in small interstitial grains; and from this on to No. 12 (inclusive) the amount of visible albite steadily increases. The jade of the Copan pebble differs markedly from the others, in texture and composition of the pyroxene; it has been described elsewhere.[4] No. 13 is the analysis of a small bead given me by Dr. Salazar, which is remarkable as being related to nephrite. It is composed of a densely felted mixture of pyroxene and zoisite, the latter as determined by Merwin.

TABLE I

ANALYSES OF MIDDLE AMERICAN JADES

	1	2	3	4	5	6	7	8	9	10	11	12	13
SiO_2	55.50	58.70	57.92	59.03	59.80	59.93	59.60	60.99	63.47	64.64	62.64	67.16	46.00
Al_2O_3	12.33	22.55	19.74	23.26	21.79	23.38	21.85	22.20	20.76	18.83	14.92	18.81	18.64
Fe_2O_3	1.41	0.48	1.08	0.41	0.20	0.44	0.48	....	1.27	0.46	0.60	0.27	0.92
FeO	1.33	0.53	0.48	0.42	0.66	0.41	0.58	0.65	.....	0.49	1.25	0.32	2.24
MgO	8.72	2.14	3.34	1.31	2.25	0.71	2.57	0.96	1.11	1.87	4.31	0.57	11.09
CaO	12.76	3.07	4.67	0.92	2.50	1.07	3.29	1.28	1.16	2.62	5.92	1.33	17.49
Na_2O	6.94	12.87	11.28	13.31	12.50	13.34	11.78	13.04	11.98	11.25	8.78	11.21	0.64
K_2O	0.25	0.15	0.25	0.31	0.27	0.29	0.20	0.21	0.34	0.23	0.23	0.22	0.08
H_2O+	0.10	0.12	0.80	0.65	0.32	0.42	.....	0.74	0.36	0.16	1.27	0.14	2.38
H_2O-	0.20	.....	....	0.09	.....	0.21	.....	....	.....	.....	....	.....	0.21
TiO_2	none	none	none	none	0.02	none	none	n.d.	n.d.	none	none	none	0.05
Cr_2O_3	none	0.08	0.17	0.26	0.05	0.09	.....	n.d.	n.d.	0.07	none	none	....
MnO	0.05	none	none	none	none	none	none	n.d.	trace	none	none	none	trace
	99.59	100.69	99.73	99.97	100.36	100.29	100.35	100.07	100.45	100.46	99.92	100.02	99.74
Sp. gr.	3.270	3.183	3.312	...	3.052	3.237	3.196	....	2.832	2.737	2.934	2.667	3.161
C°	....	17.9°	19.2°	...	23.9°	21.3°	22.8°	....	....	23°	19.4°	24.4°	24.4°
Jd	45.15	81.69	65.93	78.11	70.50	73.06	62.24	64.45	40.55	25.15	10.25	4.05	....
Di	51.51	11.54	18.72	5.27	11.58	3.48	12.01	5.60	6.64	11.10	25.78	4.80	....
Ab	1.67	3.45	12.54	13.82	15.44	20.10	19.65	28.27	51.14	63.47	63.40	90.87	....
An	1.67	3.32	2.81	2.80	2.48	3.36	6.10	1.68	1.67	0.28	0.57	0.28	....

1. Tuxtlite. Tuxtla statuette. Coarse-grained; pea-green. Washington analyst. *Proc. U. S. Nat. Mus.*, **60,** Art. 14, 1922 (5).
2. Chichen Itza. Spheroidal bead; coarse, dull white. Washington analyst.
3. Chichen Itza. Spherical bead; coarse, mottled green and white. Washington analyst.
4. Chichen Itza. Flat bead; coarse, mottled green and white. Washington analyst.
5. Chichen Itza. Long, cylindrical bead; mottled gray and green. Washington analyst.
6. Chichen Itza. Long, prismatic bead; coarse, light gray. Washington analyst.
7. Chichen Itza. Spheroidal bead; very coarse, mottled dark gray and green. Washington analyst.
8. "Jadeite," Mexico. Color ? G. W. Hawes analyst. Dana, *System of Mineralogy*, 1892, p. 370, No. 22.
9. "Jade," Mexico. Green and gray. P. T. Walden analyst. Penfield in Bishop, Jade Book, Vol. I, 1906. MgO given as 11.11 "Albite present."
10. Chichen Itza. Cylindrical bead; fine-grained, gray-green. Washington analyst.
11. Copan. Worked pebble; coarse, light pea-green. Washington analyst. *Jour. Wash. Acad.*, **12,** 1922 (p. 390).
12. Techalquilla, Mexico (from Guerrero?). Small triangular bead; fine-grained, tea-green. Washington analyst.
13. "Nephrite?" Atzacapotzalco, Mexico (from Guerrero?). Spheroidal bead; mottled sage- and tea-green. Washington analyst.

The general absence of titanium and manganese is to be noted, while chromium is present in easily determinable amounts in all the specimens in which it was looked for, except No. 1. Considerable chromium is present in some of the purely gray specimens, as No. 6), which shows no trace of green color.

The analyses are arranged in the order of decreasing pyroxene (diopside-jadeite) and increasing feldspar. The density varies quite regularly with this, but it is also influenced by the amount of jadeite relative to diopside in the pyroxene, those higher in jadeite showing the higher density. The relative amounts of jadeite and diopside vary widely and without any very marked regularity.

The refractive indices of some of the pyroxenes, determined by Dr. Merwin, are given in table II In this the relative amounts of jadeite and diopside (calculated to 100%) are given, the amount of albite (and anorthite) held in solid solution in the pyroxene being neglected. The serial progression in the values for α and γ as well as for the birefringence $\gamma - \alpha$,[6] as the pyroxene changes from nearly pure jadeite to pure diopside, is evident. There are slight irregularities at the extremes, and it would appear that the influence of the "occult" albite is simply to lower the refractive indices. This matter is to be studied further by Merwin and me.

TABLE II

Refractive Indices of the Diopside-Jadeite Series

	Jd	Di	α	γ	$\gamma - \alpha$
Jadeite (Tibet)[7]	98.68	1.32	1.655	1.667	0.012
Chichen Itza (No. 6)	95.45	4.55	1.651	1.667	0.016
Chichen Itza (No. 4)	93.69	6.31	1.650	1.668	0.018
Chichen Itza (No. 2)	87.62	12.38	1.655	1.672	0.017
Chichen Itza (No. 3)	77.89	22.11	1.660	1.678	0.018
Tuxtla (No. 1)	46.60	53.40	1.666	1.688	0.022
Copan pebble (No. 11)	28.45	71.55	1.665	1.693	0.028
Diopside (artificial)[8]	none	100.00	1.664	1.694	0.030

The feldspars show much less variation, which is not quite regular, but in general the feldspar becomes more albitic with increase of the amount of feldspar in the specimen. From what has been said, it is evident that the pyroxene of the wholly pyroxenic specimens contains considerable "occult" feldspar, both albite and anorthite, held in solid solution. No. 5 shows no feldspar in the thin section, although about 18% of $Ab_{86}An_{14}$ must be calculated as present. No. 6, however, which contains about 23.46% of similar normative feldspar, shows a few per cent present in the section. From a somewhat cursory study of the sections of the analysed specimens from this on to No. 12, it appears that the amount of visible (modal) feldspar is uniformly less than the calculated amount. It would seem, therefore, that the limit of solubility of albite (with a little anorthite) in diopside-jadeite is somewhat less than 20% by weight of the mixture. Further study of this somewhat interesting point is being undertaken.

The normative mineral relations of the Middle American jades are shown in the accompanying figure 1, in which the apices are the points for 100% (by weight, not molecular) of albite (including anorthite), jadeite, and diopside; the loci are those of the specimens whose analyses are given in table I (except No. 13). There appears to be a marked tendency to cluster about the general composition: 70 jadeite, 10 diopside, 20 albite; and the greater number are strung along and rather near the albite-jadeite side. It is noteworthy that none of them fall on this border line, indicating the constant presence of diopside; while two fall almost on the line of equal amounts of jadeite and diopside, and only one (the Copan pebble) falls well within the dominantly diopside-albite part of the triangle.

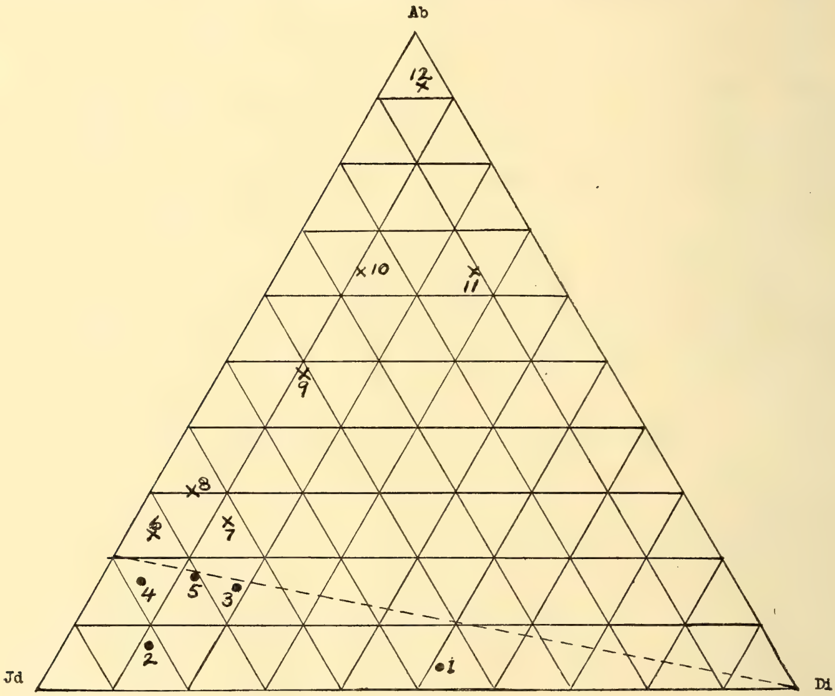

FIGURE 1
Mineral composition of jades

In figure 1 the jades without visible feldspar are indicated by dots, while those with feldspar visible in the section are shown by crosses. The areas of the two are separated by a dashed line, which extends from the diopside apex (on the basis of Bowen's experimental observation that typical diopside is incapable of holding feldspar in solution), to a point on the Ab-Jd border, at about 80 Jd–20 Ab. Some other analyses might have been plotted, which harmonize with those shown. The straightness and exact position of the dividing line are somewhat conjectural, but more jades which appear to fall near it, are being studied chemically and optically, so as to determine its position more accurately.

This series of rocks belongs evidently to the general group of jadeite jades, but differs from the usual jadeite of Burma and other sources of Chinese jade in two important particulars: the constant presence of large or considerable amounts of diopside with the jadeite in the pyroxene; and the presence of much albite in most of the series, either wholly in solid solution in the pyroxene, or partly so ("occult") and in part separately crystallized. The series, thus, extends from pure tuxtlite (diopside-jadeite, 1:1) to nearly pure albite. Further discussion of these points cannot be undertaken here, for lack of space; but it is suggested that this series shows such well-marked general and serial characters as to be deserving of a special name. That here proposed is *mayaite* which recalls the ancient nation who used and valued it, and which may distinguish it from the more widely known Burmese jadeite jades (with little or no diopside or albite). Purely jadeite jade seems to be unknown from, or at least is of very rare occurrence in, Middle America.

From these petrological considerations, the conclusion is inevitable that the material of the Central American and Mexican "jade" objects is of American and not Asian provenance. Although no American locality is yet known where *mayaite* occurs in situ, yet there are numerous specimens which are either clearly pebbles or show traces of pebble surfaces. But this is not the place for archaeological discussion. It may only be said, in conclusion, that various considerations regarding the occurrence of igneous rocks in different parts of the North American continent, as well as of the occurrence of Middle American jade objects, lead me to the belief that: (1) the jades come unquestionably from either Mexico or from Central America, or from both, and that (2) there are probably two centers of supply, both near the west (Pacific) coast, one probably in Oaxaca or Guerrero, and the other probably in Guatemala.

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[1] The term "Middle America" is used to include both Mexico and Central America.

[2] H. S. Washington, *Proc. U. S. Nat. Mus.*, **60**, Art. 14, 1922.

[3] Clarke and Merrill, *Proc. U. S. Nat. Mus.*, **11**, 1888 (122, 124).

[4] H. S. Washington, *J. Wash. Acad. Sci.*, **12**, 1922 (387).

[5] N. L. Bowen, *Amer. J. Sci.*, **40**, 1915 (161).

[6] The birefringence, γ-α, of jadeite is uniformly given in the literature as 0.029, following apparently the early determination of Lévy and Lacroix (*Les Minêraux des Roches*, 1888 (266)). This value is certainly too high for pure jadeite, and the value 0.012, determined by Merwin on an analysed and crystallographically described crystal of pure jadeite, should replace it.

[7] Merwin, in Washington, *Proc. U. S. Nat. Mus.*, **60,** Art. 14, 1922 (9).

[8] Wright and Larsen, *Amer. J. Sci.*, **37,** 1909 (33).

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*ON A COMPARISON OF THE RELATIVE SENSITIVENESS OF TELEPHONES**

By Carl Barus

Department of Physics, Brown University

Communicated, September 20, 1922

In the earlier papers I reduced the fringe deflections s, of the interferometer U-gauge, produced by telephone currents of different effective strength by the simple relation sr = const., where for a fixed effective voltage, the resistance r in the telephone circuit is very large compared with the inductance, etc. The telephone mouthpiece is connected with one or both shanks of the U-gauge by a quill tube, simple or branched, respectively, each branch containing a pinhole probe suitably directed. Thus s measures the acoustic pressure evoked when the telephone sounds.

The expression given is applicable only within the narrow range of s values and low frequencies for which it was used. For wider ranges an exponential relation, which may be written

$$\log s = \log s_0 - r/r_0$$

where s_0 and r_0 are constant, fits the observations as accurately as the fringes can be read. To give examples chosen at random for a single telephone, branched connection, salient and reëntrant pinholes (s_0 = 399 fringes, r_0 = 6380 ohms)

$r \times 10^{-3}$	4	5	6	7	8	ohms
s observed	95	65	45	32	22	fringes
s computed	94	65	46	32	22	fringes

and the same for low ranges of s(s_0 = 190, r_0 = 8620)

$r \times 10^{-3}$	8	9	11	14	18	ohms
s observed	22	17	10	5	1	fringes
s computed	22	17	10	5	2	fringes

Similarly in case of binaural telephones, one corresponding to each shank of the U-tube, $s_0 = 152$, $r_0 = 49{,}500$, the results were, for example,

$r \times 10^{-3}$	14	23	33	43	ohms
s observed	105	65	41	25	fringes
s computed	103	66	41	25	fringes

If the effective voltage is constant and $s = 1$ in the equation, $r' = r_0 \log s_0$, where r' is the resistance needed to reduce the deflection to one fringe. Hence r' is an index of the sensitiveness of telephones, all operating in the given circuit successively. A number of tests so made ranged from $r' = 3 \times 10^3$ for common telephones to $r' = 10^5$ for high resistance telephones.

The meaning of the equation given is not so easily determined, since the acoustic pressure is produced through the intermediation of the telephone mechanism. If we suppose that, within limits, the amplitude a of the air waves produced at the plate is proportional to the amplitude of the current in the telephone, the exponential equation would then belong to the acoustic pressure p, and may be written, $p = p_0 e^{a_0/a}$. This seems to me probable, as it would explain the absence of registry in case of the pinhole resonator, for instance, when the sounding organ pipe is over 5 meters distant; i.e., the rapid decline of the fringe deflection with the distance of the source of sound may thus be quantitatively evaluated.

* Advance note from a report to the Carnegie Institution of Washington, D. C.

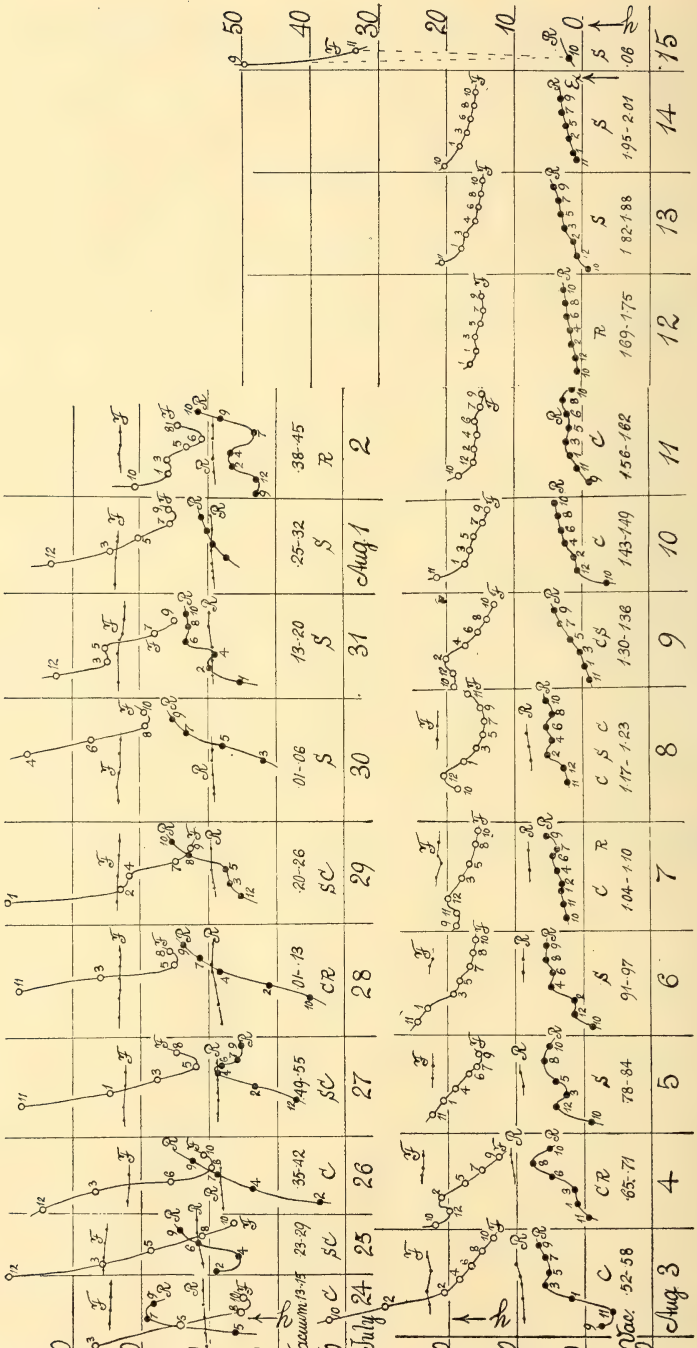
THE EQUILIBRIUM POSITIONS OF THE VACUUM GRAVITATION NEEDLE IN 1921 AND 1922*

By CARL BARUS

DEPARTMENT OF PHYSICS, BROWN UNIVERSITY

Communicated, September 20, 1922

The apparatus used last year was identical with the present one, excepting only the minor appurtenances, added to secure greater convenience in exhaustion. These are without influence. Moreover neither the location of the apparatus nor the mechanism of the weights M had been changed. The method of observation was substantially the same. It is thus very difficult to account for the erratic behavior of the gravitation needle during July and August of this year, as compared with the nearly constant deflections (Δy, y being the telescopic scale reading in centimeters) obtained under like conditions last year. I have, therefore, in the figure, put the static elongations y for the same day in 1921 and


30
20
10
0
-10
h
F
R
Vacuum 13-15 C
July 24
23-29 SC
25
35-42 C
26
.49-.55 SC
27
.01-.13 CR
28
.20-.26 SC
29
.01-.06 S
30
.13-.20 S
31
.25-.32 S
Aug. 1
.38-.45 R
2
50
40
30
20
10
0
Vac. .52-.58 C
Aug 3
.65-.71 CR
4
.78-.84 S
5
.91-.97 S
6
1.04-1.10 C R
7
1.17-1.23 C S C
8
1.30-1.36 CS
9
1.43-1.49 C
10
1.56-1.62 C
11
1.69-1.75 R
12
1.82-1.88 S
13
1.95-2.01 S
14
.06 S
15
E

1922 together, successive observations of the position of equilibrium (about one hour apart) being inserted equidistant horizontally. Data for 1922 are distinguished by little circles with the nearest hour number of the observation attached. Data for 1921 are given in points, in corresponding positions, for easy comparison. F denotes that the attracting weight M on the right end of the needle is in front of it. R, that M is to the rear of the needle. Complete sets of observations (i.e. equilibrium curves F and R) are given between about 10 A.M. and 10 P.M. for each year from July 24 to August 8 and to Aug. 15 in 1922. The exhaustion of the case in the morning and at night is shown on each day for 1922 only, as the data were not taken in 1921. In the same place C refers to cloudy, C' to partly cloudy weather, S to sunshine or a clear day and R to rain. It is seen at a glance that the variations of the position of equilibrium y in the lapse of time, are of a different order in 1922, from their approximate constancy in the given scale, in 1921. The contrast is startling with nothing easily apparent to account for it, unless it is traceable to a difference in the respective vacua. For the exhaustions, as a whole, were thought to have been of about the same order of value in both years. In 1922 there was fresh exhaustion on July 24, 28 and 30. Thereafter I left the apparatus with its slight leak (12 mm. per day), to itself, until August 15. In 1921 there were no intermediate exhaustions and unfortunately the McLeod gauge was not attached to the case.

The figure shows that all observations have a period of one day (24hr). Consequently the variations cannot be contributed by anything within the laboratory, though their enhancement is possible. They must in other words be originally meteorological, and due to solar radiation. Immediately after exhaustion (as on July 28, 30) there are apt to be large deflections, probably showing that the air within the case, though very rare, has nevertheless an appreciable cooling or other effect. For this reason I ceased to make exhaustions after July 30, and the observations, possibly for this contributory reason among others, gradually grew smoother and more normal. Any slight cooling of the inside of the case would tend to exaggerate the day time meteorological effect. If the cooling due to an exhaustion from 1 mm. to .001 mm. of pressure were instantaneous one would estimate that about .01 ca. would be abstracted from the air within the case. This is largely supplied of course by the massive case and the exhaustion is necessarily very gradual. Nevertheless the effect on the needle is always very marked. One cannot observe with the pump running, as the needle persistently clings to the wall of the case.

An inspection of the equilibrium curves in July, 1922, shows that F and R are quite liable to intersect somewhere between 6 and 8 P.M. which means that the radiant forces are in excess (Δy negative) and they may thereafter considerably overbalance the gravitational pull. This happens when the temperature-time coefficient $d\theta/dt$ is negative. When the latter is positive, gravitation and the radiant forces coöperate. It is seen, moreover, that the higher exhaustions (a few tenths mm. and less), are (like the plenum) favorable to strong radiant forces; whereas in moderate exhaustions (a few millimeters) the radiant forces tend to a minimum, so that the night observations become more and more trustworthy. This may be inferred from the gradual simplification of the *F*, *R* curves between Aug. 1 and August 14. In corroboration, the apparatus was again exhausted on Aug. 14 and tested about 12 hours later on the morning of Aug. 15. It is seen that the strong radiant forces have reappeared.

* Advance note from a report to the Carnegie Institution of Washington, D. C.

FURTHER MEASUREMENTS OF STELLAR TEMPERATURES AND PLANETARY RADIATION

BY W. W. COBLENTZ

BUREAU OF STANDARDS, WASHINGTON, D. C.

Communicated September 21, 1922

1. *Introductory Statement.*—In a previous communication[1] estimates were given of the temperatures of 16 stars as determined from their spectral energy distribution which was determined by means of a new spectral radiometer, consisting of a series of transmission screens and a vacuum thermocouple.

By means of these screens, which, either singly or in combination, had a uniformly high transmission over a fairly narrow region of the spectrum and terminating abruptly to complete opacity in the rest of the spectrum, it was possible to obtain the radiation intensity in the complete stellar spectrum as transmitted by our atmosphere.

Not being equipped at that time for making radiometric measurements on the Sun, the effective temperature of which is known with some degree of accuracy, and hence could be used as a standard, an estimate of the effective temperature of a star was obtained by two methods, one of which consisted in using a solar type star (α Aurigae, Class G o) as a standard. This seemed permissible in view of the fact that the observed temperature (6000° K) of α Aurigae was found to be in close agreement with that assigned to the Sun.

The object of the present communication is to report a verification of the above-mentioned stellar radiation measurements, by similar measurements recently made on the Sun; also to give some measurements of the radiation from the major planets.

2. *Apparatus and Procedure.*—The apparatus used in this work consisted of the 40-inch reflector of the Lowell Observatory, to which was attached a special radiometer mentioned in a previous issue of these PROCEEDINGS,[1] and fully described elsewhere.[2] The thermocouples were new and were made of fine bismuth wire, with receivers 0.2 mm. in diameter.

The crucial part of this investigation consisted in obtaining an exact measurement of the water cell transmission of the direct sunlight. For this purpose, the great reflecting mirror and the end of the telescope tube were covered. Parallel light, entering a small opening in the cover of the telescope tube, fell upon a small, plane, silvered mirror, placed in the axis of the tube, and from thence reflected upon the diagonal mirror to a thermopile which was especially constructed for these measurements.

3. *Experimental Data.*—The solar radiation data are given in table 1, in which column 1 gives the spectral range, and columns 2 and 4 give the spectral radiation components (A.M. and P.M.) of the Sun (also of α Aurigae, column 6) corrected for all losses except atmospheric absorption. The corresponding temperatures are given in columns 3, 5 and 7.

TABLE 1

SPECTRAL RADIATION COMPONENT OF THE SUN AND OF α AURIGAE IN PER CENT OF THE TOTAL

	SUN				α AURIGAE	
	A.M.		P.M.			
Spectral range. μ = 0.001 mm.	Spectral radiation component	Temperature °K	Spectral radiation component	Temperature °K	Spectral radiation component	Temperature °K
0.30μ to 0.43μ	25.0%	5670	25.6%	5750	18.4%	5000?
0.43μ to 0.60μ	22.1	6500	19.0	5760	18.2	5800
0.60μ to 1.40μ	31.8	5860	35.0	5050?	30.6	6000
1.40μ to 4.10μ	20.6	6060	20.1	6140	26.4	5200?
4.10μ to 10.00μ	0.5	....	0.8	....	6.4	....
Average		6000° K		5900° K		5900° K

These data are interesting in showing a close agreement between the spectral radiation components of the Sun and α Aurigae in the visible spectrum (0.43μ to 0.6μ) which is in agreement with the visual and photographic observations. However, in the ultra-violet the radiation component of α Aurigae is less, and in the infra-red it is greater (owing partly to the low-temperature radiation from the companion star), than that of the Sun. The lower value in the ultra-violet might be owing, in part, to the larger air mass traversed and hence to the greater amount of scattering of the incoming radiations. But this does not explain, and is inconsistent with, the higher values observed in the infra-red, in many of the stars previously measured, in spite of the greater air mass traversed.

No doubt this is owing partly to experimental errors; but some of it may be attributed to the low (selective) infra-red emission of the Sun, the earth's atmosphere being quite transparent to radiations greater than 7 μ.

For the purposes of this investigation the agreement between the observed temperatures of α Aurigae and the Sun is considered satisfactory, verifying the previous measurements of stellar temperatures which range from 3000° K for red, class-M stars, to 12,000° K for blue, class-B stars.

Planetary Radiation.—The thermal radiation emitted from a planet as a result of warming by exposure to solar radiation, including heat which may be radiated by virtue of a possible high internal temperature of the planet itself, is essentially of long wave-lengths 7 μ to 12 μ. Hence, by means of a 1 cm. cell of water, interposed in the path of the total radiation emanating from the planet, this long wave-length radiation can be separated from the reflected solar radiation, and in this manner a measurement obtained of the energy reradiated. If there is planetary radiation, the water cell transmission will be less than that of the direct solar radiation.

The observed transmission of planetary radiation through a cell of water 1 cm. in thickness is given in table 2. This table shows that the water cell transmission of the total radiation from Jupiter is practically the same as that of the direct solar radiation. From this it appears that the outer atmosphere of Jupiter does not radiate appreciable energy as the result of warming by solar rays and that the atmosphere is sufficiently thick and opaque to trap all the energy reradiated as the result of warming of its interior by solar radiation, or by internal heating.

TABLE 2

TRANSMISSION OF PLANETARY RADIATION THROUGH A CELL OF WATER 1 CM. IN THICKNESS; ALSO THE PER CENT OF PLANETARY RADIATION EMITTED

			PER CENT TRANSMISSION			
Object	Date June, 1922 MST	No. of sets of measurements	Deviation from mean	Mean value 1922	Mean value 1914	Planetary radiation in per cent of the total
Sun	17th; 11:30 A.M.	5	1.2%	69.3	....	..
	4:15 P.M.	4	0.5	69.8	....	..
Jupiter						
Central disk	14th; 10:15 P.M.	2	1.4	69.7	65.6	..
Southern hemisphere	10:20 P.M.	2	3.0	66.8	65.8	..
Venus	15th; 8:00 P.M.	2	2.3	66.3	59.0	5
Saturn	14th; 9:00 P.M.	3	1.1	60.0	55	15
Mars	15th; and 18th	.	6.0	50.3	....	30
Moon	....	..	....	....	14.7	80

The radiometric measurements on Venus, Jupiter and Saturn are in good agreement with similar measurements made at Mt. Hamilton, Calif., in 1914, showing a decidedly lower transmission of radiation through the water cell, in the case of Venus and Saturn.

The intensity of the planetary radiation increases with decrease in the density of the surrounding atmosphere and (as interpreted from the water cell transmissions) in per cent of the total radiation emitted, is as follows: Jupiter (O), Venus (5), Saturn (15), Mars (30) and the Moon (80).

The water cell transmission of the radiations from the southern (50.6%) and northern (53.1%) hemispheres of Mars should be and is higher than that of the radiations emanating from the equatorial (47.3%) region, owing to the depletion of the reradiated energy by the greater air mass. Moreover, the intensity of the planetary radiation from the northern hemisphere of Mars was found to be less than from the southern hemisphere. This is to be expected in view of the observed cloudiness over the northern hemisphere which is usually the brighter and is approaching the winter season, and hence is at a lower superficial temperature.

These data were obtained through the generosity of the Lowell Observatory, Flagstaff, Arizona, who financed this research. Dr. C. O. Lampland again kindly operated the telescope and it is intended to publish the complete results in a joint paper on the measurements of planetary radiation and their astronomical significance.

As already stated, the object of the present note is primarily to place on record a verification of the results, obtained a year ago, on stellar temperatures.

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[1] *Proc. Nat. Acad. Sci.*, **8**, 1922 (49–53).

[2] *Bureau of Standards Scientific Paper*, No. **438**, 1922.

PROOF OF A THEOREM DUE TO HEAVISIDE[1]

By H. A. Lorentz

Haarlem, Holland

Communicated August 7, 1922

In the *Electrical Papers*, Vol. 2, p. 412, Heaviside states: "The whole work done by impressed forces suddenly started exceeds the amount representing the waste by Joule-heating at the final rate (when there is any), supposed to start at once, by twice the excess of the electric over the magnetic energy of the steady field set up."

Consider a system of bodies, either conductors or dielectrics, or, for the sake of generality, both at the same time, and denote by **E** the electric force, by **H** the magnetic force, by **D** the dielectric displacement, by **C** the current of conduction, by **B** the magnetic induction. We shall suppose the matter to be isotropic, so that at each point we have to introduce only one ϵ (dielectric constant), one permeability μ, and one conductivity σ.[2] These quantities may change from place to place; they, as well as **E, H,** etc., are considered as continuous functions. If there are different bodies, transition layers in which the properties gradually change from one body to the other, are supposed to exist at the common boundary. (This is only for the purpose of mathematical convenience; the thickness of the transition layer can be supposed to diminish indefinitely, and so we can pass to the limiting case of a sharp demarcation.) Let there be two kinds of impressed forces ("electromotive" forces), the one $\mathbf{F}_1$ producing dielectric displacement, the other $\mathbf{F}_2$ producing conduction currents.[3] The meaning of this is that the dielectric displacement is determined by

$$\mathbf{D} = \epsilon(\mathbf{E} + \mathbf{F}_1) \qquad (1)$$

and the conduction current by

$$\mathbf{C} = \sigma(\mathbf{E} + \mathbf{F}_2) \qquad (2)$$

The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are again continuous functions of the coördinates. Each of them may be confined to a limited space and these two spaces may lie one outside the other or they may overlap more or less. At a definite point there is but one **E**, the same in (1) and (2).

In addition to (1) and (2) we have

$$\mathbf{B} = \mu\mathbf{H} \qquad (3)$$

$$\text{curl}\,\mathbf{H} = \frac{1}{c}(\dot{\mathbf{D}} + \dot{\mathbf{C}}) \qquad (4)$$

$$\text{curl}\,\mathbf{E} - \frac{1}{c}\dot{\mathbf{B}} \qquad (5)$$

$$\text{div}\,(\dot{\mathbf{D}} + \mathbf{C}) = 0 \qquad (6)$$

$$\text{div}\,\mathbf{B} = 0 \qquad (7)$$

Electric energy per unit of volume $\mathbf{D}^2/2\epsilon$.
Magnetic energy per unit of volume[4] $^1/_2(\mathbf{B}.\mathbf{H})$.
Joule-heat per unit of volume and unit of time $\mathbf{C}^2/\sigma$.
Work of $\mathbf{F}_1$ per unit of volume and unit of time $(\mathbf{F}_1 \cdot \dot{\mathbf{D}})$.
Work of $\mathbf{F}_2$ per unit of volume and unit of time $(\mathbf{F}_2\ \mathbf{C})$.

We shall suppose that the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are started during an infinitely short interval of time Δt, ending at the instant $t = 0$; from this latter onward they remain constant. Finally there will be a steady state. In what follows the integrations with respect to the time are extended up to an instant t at which the final state has been practically reached. The final values of **D**, **C**, etc. are denoted by $\overline{\mathbf{D}}$, $\overline{\mathbf{C}}$, etc.

The main point in the following reasoning is that, when the interval Δt is infinitely short, the values of **D** and **B** at the instant $t = 0$ are likewise infinitely small, so that we may reckon with $\mathbf{D} = 0$, $\mathbf{B} = 0$ for $t = 0$; moreover, at $t = 0$ the current **C** cannot be infinite and we may neglect the work of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ during the interval Δt. In the time-integral representing the work (up to the time t), we may take $t = 0$ for the lower limit.

Proof.—During the interval Δt the finite forces $\mathbf{F}_1$ and $\mathbf{F}_2$ cannot produce an infinite **E**. Therefore, according to (5) $\dot{\mathbf{B}}$ can at the best have a finite value. But then (because at the beginning of Δt, $\mathbf{B} = 0$) **B** must be infinitely small until $t = 0$. The same is true of **H** (because connected with **B** by (3)). Therefore, on account of (4), $\dot{\mathbf{D}} + \mathbf{C}$ must be infinitely small. But, as (2) shows, **C** cannot be infinite; therefore $\dot{\mathbf{D}}$ also cannot be infinite, and **D** must be infinitely small at $t = 0$.[5] We can conclude now that both the integrals $\int(\mathbf{F}_1 . \dot{\mathbf{D}})dt$ and $\int(\mathbf{F}_2 . \mathbf{C})dt$ which represent the work of the electromotive forces, when taken over the infinitely small time Δt, have infinitely small values; indeed, neither $(\mathbf{F}_1 . \dot{\mathbf{D}})$ nor $(\mathbf{F}_2 . \mathbf{C})$ becomes infinite in the interval.

dS is the element of volume. Integrations with respect to dS are extended over the whole field, when necessary to infinite distance.

The electric energy in the final state is $U = {}^1/_2 \int \frac{\overline{\mathbf{D}}^2}{\epsilon} d\,S$, or, on account of (1)[6]

$$U = \frac{1}{2}\int(\{\overline{\mathbf{E}} + \mathbf{F}_1\} . \overline{\mathbf{D}})dS \qquad (8)$$

The magnetic energy in the final state is $T = \frac{1}{2}(\overline{\mathbf{H}} . \overline{\mathbf{B}})dS$.

The Joule heat per unit of time in the final state is

$$\int \frac{\overline{\mathbf{C}}^2}{\sigma} dS,$$

for which we may also write[7] $\int(\mathbf{F}_2 . \overline{\mathbf{C}})dS$.

If the Joule generation of heat started suddenly at $t = 0$ at the rate given by the last expression, the amount of heat developed until the time t would be[8]

$$W = \int_0^t dt \int (\mathbf{F}_2 . \overline{\mathbf{C}})dS \qquad (9)$$

The work of the impressed forces up to the time t is given by

$$A = \int_0^t dt \int (\mathbf{F}_1 . \dot{\mathbf{D}})dS + \int_0^t dt \int (\mathbf{F}_2 . \mathbf{C})dS, \qquad (10)$$

where we may write for the first term $\int(\mathbf{F}_1.\bar{\mathbf{D}})dS$ because $\mathbf{F}_1$ is constant and $\int_0^t \dot{\mathbf{D}}dt = \bar{\mathbf{D}}$($\mathbf{D}$ being 0 for $t = 0$).

From (10) and (9):

$$A - W = \int(\mathbf{F}_1.\bar{\mathbf{D}})dS + \int_0^t dt \int(\mathbf{F}_2.\{\mathbf{C} - \bar{\mathbf{C}}\})dS \tag{11}$$

In virtue of (8):

$$\int(\mathbf{F}_1.\bar{\mathbf{D}})dS = 2U - \int(\bar{\mathbf{E}}.\bar{\mathbf{D}})dS,$$

and by (2), applied to the final state, $\mathbf{F}_2 = \bar{\mathbf{C}}/\sigma - \bar{\mathbf{E}}$. Substituting these values in (11) we find

$$\begin{aligned} A - W = 2U + \int_0^t dt \int\left(\frac{1}{\sigma}\bar{\mathbf{C}}.\{\mathbf{C} - \bar{\mathbf{C}}\}\right)dS - \\ - \int(\bar{\mathbf{E}}.\bar{\mathbf{D}})dS - \int_0^t dt \int(\bar{\mathbf{E}}.\{\mathbf{C} - \bar{\mathbf{C}}\})dS \end{aligned} \tag{12}$$

Now, we may write

$$\bar{\mathbf{D}} = \int_0^t \dot{\mathbf{D}}dt,$$

and we can therefore combine the two last terms of (12) into

$$- \int_0^t dt \int(\bar{\mathbf{E}}.\{\dot{\mathbf{D}} + \mathbf{C} - \bar{\mathbf{C}}\})dS \tag{13}$$

We have by (5) and (6)

$$\text{curl}\,\bar{\mathbf{E}} = 0, \qquad \text{div}\,(\dot{\mathbf{D}} + \mathbf{C}) = 0, \qquad \text{div}\,\bar{\mathbf{C}} = 0;$$

hence (13) vanishes[9] and (12) reduces to

$$A - W = 2U + \int_0^t dt \int\left(\frac{1}{\sigma}\bar{\mathbf{C}}.\{\mathbf{C} - \bar{\mathbf{C}}\}\right)dS \tag{14}$$

Transformation of the last term: Substitute $\bar{\mathbf{C}} = c$ curl $\bar{\mathbf{H}}$ (form. (4) applied to final state), $\mathbf{C} - \bar{\mathbf{C}} = \sigma(\mathbf{E} - \bar{\mathbf{E}})$ (form. (2) combined with corresponding equation for final state), integrate by parts, and finally replace curl $\mathbf{E}$ by $-\dot{\mathbf{B}}/c$ and curl $\bar{\mathbf{E}}$ by 0.

$$\begin{aligned} \int_0^t dt \int\left(\frac{1}{\sigma}\bar{\mathbf{C}}.\{\mathbf{C} - \bar{\mathbf{C}}\}\right)dS &= \int_0^t dt \int (c\,\text{curl}\,\bar{\mathbf{H}}.\{\mathbf{E} - \bar{\mathbf{E}}\})\,dS \; \text{[10]} \\ = \int_0^t dt \int(c\bar{\mathbf{H}}.\text{curl}\,\{\mathbf{E} - \bar{\mathbf{E}}\})dS &= - \int_0^t dt \int (\bar{\mathbf{H}}.\dot{\mathbf{B}})dS \\ &= - \int dS \int_0^t (\bar{\mathbf{H}}.\dot{\mathbf{B}})dt \end{aligned} \tag{15}$$

Now, since $\overline{\mathbf{H}}$ does not depend on t, and $\mathbf{B} = 0$ for $t = 0$,

$$\int_0^t (\overline{\mathbf{H}}.\dot{\mathbf{B}})dt = (\overline{\mathbf{H}}.\overline{\mathbf{B}});$$

(15) therefore becomes

$$-\int (\overline{\mathbf{H}}.\overline{\mathbf{B}})dS = -2\,T$$

and (14)

$$A - W = 2(U - T) \qquad \text{q. e. d.} \tag{16}$$

Remarks.—1. Both A and W depend on the choice of the time t the end of the (long) interval of time which we considered. But the *difference* $A - W$ is independent of that choice. Indeed, in the steady state, the work of the impressed force per unit of time is equal to the generation of heat, equally per unit of time, so that, when the interval of time is lengthened, A and W increase by equal amounts.

When, in the final state, there are no currents (but only electric charges) $W = 0$. In this case A is independent of the choice of t.

2. W is not the quantity of heat that is *really* generated. This latter quantity W' is given by

$$W' = \int_0^t dt \int \frac{\mathbf{C}^2}{\sigma} dS.$$

3. In the interval of time during which the steady state is reached there will in general be a *radiation* of energy from the system outward. Let the total amount of the radiated energy (to be calculated by applying Poynting's rule to the flow through an infinitely great sphere) be R. Then, by the law of conservation of energy

$$A = W' + U + T + R.$$

Comparing this with (16) one finds

$$W - W' = 3T - U + R.$$

4. The theorem expressed by (16) can easily be verified for simple cases, when all quantities involved can be completely calculated.

(a). A linear circuit with resistance r and self-inductance L, no capacity. Let the electromotive force $\mathbf{E}$ be suddenly applied at $t = 0$, remaining constant ever afterwards

The current is given by $i = E/r.(1 - e^{-rt/L})$, and one finds

$$A = \int_0^t Eidt = \frac{E^2}{r}t - \frac{E^2L}{r^2} \quad (t \text{ very great}),$$

$$W = \frac{E^2}{r}t, \quad T = \frac{LE^2}{2r^2}, \quad U = 0.$$

(*b*). A dielectric with the constant ϵ fills all space. At time $t = 0$ an electromotive force $\mathbf{F}_1$ is suddenly applied. It has the same direction and magnitude at all points inside a sphere and is limited to that part of space. The steady state is now easily calculated; likewise the amount of energy R, that is radiated.

5. The theorem mentioned in G. W. Pierce, *Electrical Oscillations*, p. 40, is closely connected with Heaviside's general theorem.

Consider a circuit containing a condenser and in series with it a self-induction and resistance. In this case $T = 0$, $W = 0$, so that (16) becomes $A = 2U$. But, by the law of energy ($R = 0$), $A = W' + U$. Therefore $W' = U$, which is Pierce's statement.

---

[1] This general theorem stated without proof by Heaviside, was useful to Mr. Charles Manneback in his thesis for the doctorate in science at the Massachusetts Institute of Technology, June 1922: *Radiation from Transmission Lines* (as yet unpublished). As Dr. Manneback could not locate, in the textbook or periodical literature, a satisfactory proof of the theorem in all its generality, he appealed for help to Professor Lorentz who kindly sent the demonstration. It has seemed as though the treatment of the theorem by Dr. Lorentz with his comments upon it might be of wide interest, particularly on account of possible practical applications, and he has assented to its publication.—EDITOR.

[2] One can just as well (somewhat greater complication of the formulae) consider crystalline substances.

[3] One could add a "magneto-motive" force, but for the sake of simplicity we shall not do so.

[4] $(\mathbf{A}.\mathbf{B})$ is the scalar product of the vectors $\mathbf{A}$ and $\mathbf{B}$; $(\mathbf{A}.[\mathbf{B} + \mathbf{C}])$ the scalar product of the vectors $\mathbf{A}$ and $\mathbf{B} + \mathbf{C}$. Of course when two vectors have the same direction (as $\mathbf{B}$ and $\mathbf{H}$) we can as well take the product of their magnitudes.

[5] What is said here of $\mathbf{D}$ does not apply to the electric force. When, during the interval Δt, $\mathbf{D}$ is infinitely small (practically 0), *eq.* (1) shows that $\mathbf{E}$ is nearly $-\mathbf{F}_1$, a finite value. Note that we have drawn our conclusions from the fact that the equations contain $\dot{\mathbf{B}}$ and $\dot{\mathbf{D}}$. They do not contain $\dot{\mathbf{E}}$.

[6] We need not write $\overline{\mathbf{F}}_1$ and $\overline{\mathbf{F}}_2$, because $\mathbf{F}_1$ and $\mathbf{F}_2$ are constant from $t = 0$ onward.

[7] This formula expresses the fact that in the final state, when both U and T remain constant, the Joule heat is equal to the work of $\mathbf{F}_2$. (The work of $\mathbf{F}_1$ is 0, because in the final state $\dot{\mathbf{D}} = 0$.) One can deduce one form from the other by using $\overline{\mathbf{C}} = \sigma\,(\overline{\mathbf{E}} + \mathbf{F}_2)$. $\int (\overline{\mathbf{E}}.\overline{\mathbf{C}})dS = 0$, because div $\overline{\mathbf{C}} = 0$ and curl $\overline{\mathbf{E}} = 0$.

[8] Since $(\mathbf{F}_2.\overline{\mathbf{C}})$ is independent of t we could just as well write $t.\int (\mathbf{F}_2.\overline{\mathbf{C}})dS$, but in what follows, the form (9) is to be preferred.

[9] First extend the integration to the space within a sphere Σ around O, the radius r of which ultimately increases indefinitely. Since $\overline{\mathbf{E}}$ depends on a potential φ, we find by partial integration (putting $\dot{\mathbf{D}} + \mathbf{C} - \overline{\mathbf{C}} = \mathbf{Q}$) $\int (\overline{\mathbf{E}}.\mathbf{Q})dS = -\int (\text{grad}\,\varphi.\mathbf{Q})dS = -\int \varphi \mathbf{Q}_n d\Sigma + \int \varphi\, \text{div}\, \mathbf{Q} dS$. ($\mathbf{Q}_n$ normal component.) The surface integral vanishes for $r = \infty$ because (when the total charge of the system is 0) φ decreases at least as $1/r^2$, whereas $\mathbf{Q}_n$ becomes 0.

[10] There the integration by parts leads to the surface integral $c\int \{[\overline{\mathbf{H}}_{y_2}(\mathbf{E}_z - \overline{\mathbf{E}}_z) - \overline{\mathbf{H}}_z(\mathbf{E}_y - \overline{\mathbf{E}}_y)]\cos\alpha + \ldots\} d\Sigma$(cos α ....direction constants of normal) to Σ and this vanishes for similar reasons as the surface integral of the preceding note.

THE STABILITY OF THE NORMAL AGE DISTRIBUTION[1]

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SCHOOL OF HYGIENE AND PUBLIC HEALTH, JOHNS HOPKINS UNIVERSITY

Communicated, September 13, 1922

There is a unique age distribution which, in certain circumstances,[2] has the property of perpetuating itself when once set up in a population. This fact is easily established,[3] as is also the analytical form of this unique *fixed* or *normal* age distribution.

More difficult is the demonstration that this age distribution is *stable*, that a population will spontaneously revert to it after displacement therefrom, or will converge to it from an arbitrary initial age distribution. Such a demonstration has hitherto been offered only for the case of small displacements,[4] by a method making use of integral equations. The purpose of the present communication is to offer a proof of stability which employs only elementary analytical operations, and which is readily extended to cover also the case of large displacements. This method presents the further advantage that it is molded in more immediate and clearly recognizable relation to the physical causes that operate to bring about the normal age distribution.

Consider a population which, at time t has a given age distribution such as that represented by the heavily drawn curve in fig. 1, in which the abscissae represent ages a (in years, say), while the ordinates y are such that the area comprised between two ordinates erected at a_1 and a_2, respectively, represents the number of individuals between the ages a_1 and a_2.

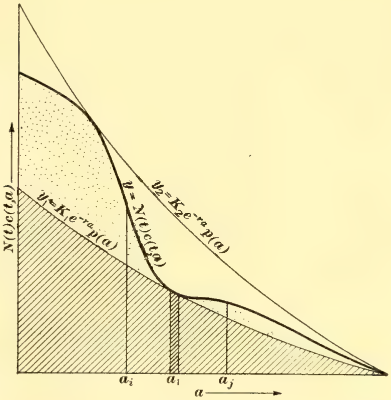

Fig. 1

If we denote by $N(t)$ the total population at time t, and if the ordinates of our curve are

$$y = N(t)\ c(t,a) \tag{1}$$

we have, evidently,

$$\int_{a_1}^{a_2} y\ da = N(t) \int_{a_1}^{a_2} c(t,a)da = N(t,a_1,a_2) \tag{2}$$

where $N(t,a_1,a_2)$ denotes the number of individuals living at time t and comprised within the age limits a_1 and a_2. We may speak of $c(t,a)$ as the coefficient of age distribution. It is, of course, in general a function of t, only in the special case of the fixed or self-perpetuating age distribution is $c(a)$ independent of the time.

Now, without assuming anything regarding the stability of the self-perpetuating age distribution, it is easy to show[2] that its form must be

$$c(a) = \frac{e^{-ra}p(a)}{\int_0^{\infty} e^{-ra}p(a)da} = be^{-ra}p(a) \tag{3}$$

where r is the real root of the equation

$$1 = \int_{a_i}^{a_j} e^{-ra}p(a)\beta(a)da \tag{4}$$

In this equation r is the natural rate of increase of the population, i.e., the difference $r = b - d$ between the birthrate per head b and the deathrate per head d, and $p(a)$ is the probability, at birth, that a random individual will live to age a (in other words, it is the principal function tabulated in life tables, and there commonly denoted by 1_x). The limits a_i and a_j of the integral are the lower and upper age limits of the reproductive period. The factor $\beta(a)$, which might be termed the procreation factor, or more briefly the birth factor, is the average number of births contributed per annum by a parent of age a. (In a population of mixed sexes it is, of course, immaterial, numerically, to what parent each birth is credited. It will simplify the reasoning, however, if we think of each birth as credited to the female parent only.)

The factor $\beta(a)$ will in general itself depend on the prevailing age distribution. This is most easily seen in the case of extremes, as for example in a population which should consist exclusively of males under one year of age and females over 45. But, except in such extreme cases, $\beta(a)$ will not vary greatly with changes in the age constitution of the population, and we shall first develop our argument on the supposition that $\beta(a)$ is independent of the age distribution. We shall then extend our reasoning to the more general case of $\beta(a)$ variable with $c(t,a)$.

Referring now again to fig. 1, let two auxiliary curves be drawn, a *minor tangent curve* and a *major tangent curve*

$$y_1 = K_1e^{-ra}p(a) \quad (5) \qquad y_2 = K_2e^{-ra}p(a) \quad (6)$$

the constants K_1, K_2 being so chosen that the minor tangent curve lies wholly beneath the given arbitrary curve, except where it is tangent thereto, while the major tangent curve lies wholly above the given curve, except where it is tangent thereto.

The given arbitrary curve representing the age constitution of the population at time t then lies wholly within the strip or area enclosed between the minor and the major tangent curves.

Now consider the state of affairs at some subsequent instant t'. Had the population at time t consisted solely of the individuals represented by the lightly shaded area in fig. 1, i.e. the area under the minor tangent curve, then at time t' the population would be represented by the lower curve of fig. 2, whose equation is

$$y'_1 = K'_1 e^{-ra} p(a) \qquad (7)$$

$$= K_1 e^{r(t'-t)} e^{-ra} p(a) \qquad (8)$$

For the age distribution (5) is of the fixed form (3), and therefore persists in (7); on the other hand, given such fixed age distribution, the population as a whole increases in geometric progression,[3] so that $K'_1 = K_1 e^{r(t'-t)}$. In point of fact, we have left out of reckoning that portion of the population which in fig. 1 is represented by the dotted area. Hence, in addition to the population under the lower curve of fig. 2, there will, at time t', be living a body of population which for our present purposes it is not necessary to determine numerically. We need only know that it is some positive number, so that the curve representing the actual population at time t' must lie wholly above or in contact with the curve (8).

By precisely similar reasoning it is readily shown that at time t' the actual curve lies wholly beneath or in contact with the curve

$$y'_2 = K_2 e^{r(t'-t)} e^{-ra} p(a) \qquad (9)$$

Hence at time t' the actual curve lies wholly within the strip comprised between the two curves (8) and (9).

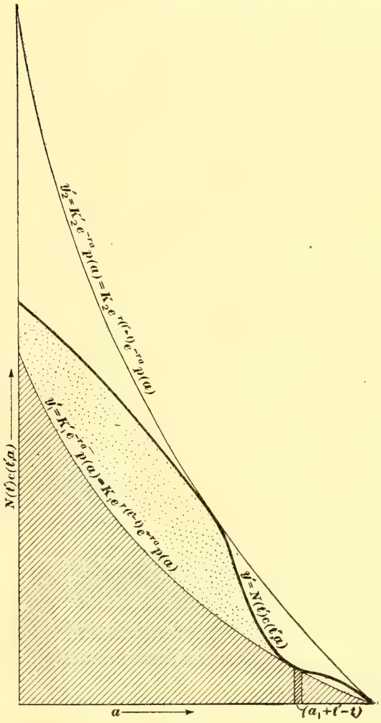

Fig. 2

The curves shown are intended to be interpreted only in a qualitative sense; so, for example, the increase in the ordinates in passing from fig. 1 to fig. 2 is very much exaggerated, to render it obvious to the eye.

Consider now an elementary strip, of width da, of the original population (shown heavily shaded in fig. 1), which at time t is in contact with the minor tangent curve. Let this contact occur at age a_1, so that $y_t(a_1) = K_1 e^{-ra_1} p(a_1)$. At time t' the survivors of the individuals comprised in this elementary strip will be of age $(a_1 + t' - t)$, so that they will then be represented by a strip of width da and of altitude

$$\begin{aligned} y_{t'}(a_1 + t' - t) &= K'_1 e^{-ra_1} p(a_1 + t' - t) \\ &= K_1 e^{r(t'-t)} e^{-r(a_1 + t' - t)} p(a_1 + t' - t) \end{aligned}$$

From this it is seen that the elementary strip of population which at time t' contacts with the minor tangent curve (8) is built up of the survivors of the strip which at time t contacted with the minor tangent curve (5). In other words, if we follow up, by identity of individuals, the element of the population which at any instant contacts with the minor tangent curve, this element (so long as any part of it survives) continues in contact with that curve. (It must be remembered, however, that the tangent curve itself changes with time according to (5), (8).)

Similarly it follows that the element of population which at any instant contacts with the major tangent curve continues in this contact so long as any part of it survives.

And again, considering any element of the population which does not contact with the minor or the major tangent curves, but has its upper extremity at some point within the area enclosed between these curves, it can be shown by precisely similar reasoning that such element continues in such intermediate position.

Turning now from the consideration of the survivors of the original population, and taking in view the new population added by births since the time $t = t$, we note that if the original population had been that represented by the shaded area in fig. 1, i.e., by the area under the minor tangent curve, then the birthrate would at all times have been such as continuously to reproduce a population represented by the minor tangent curve (5), (8).

In point of fact, provided that contact with the minor tangent curve is not *continuous* over a range of ages equal to or greater than $a_j - a_i$, it is easily seen that the total birthrate

$$B = N \int_{a_i}^{a_j} c(a)\beta(a)da$$

is always greater (equality is here excluded) than that which would result from and in turn reproduce the age distribution represented by the minor tangent curve.

Similarly the total birthrate is always less than that which would result from a population and age distribution represented by the major tangent curve.

From this it is clear that, after the original population has once died out, the representative curve can never again contact with the original minor and major tangent curves, but must henceforth be separated from them everywhere by a finite margin (except, of course, where the several curves terminate upon the axis of a).

We may then begin afresh by drawing a new pair of tangent curves, lying within the original pair, and so on indefinitely, until the minor and major tangent curves coincide, and with them also coincides the actual curve of age distribution, which is then of the form

$$N(t)\ c(a) = Ke^{rt}e^{-ra}p(a)$$

which we recognize as the fixed age distribution (3), with

$$N(t) = Ke^{rt}\int_0^\infty e^{-ra}p(a)da$$

In this argument we have expressly excluded the case that the original age distribution curve contacts *continuously* with one of its tangent curves over a range greater than the reproductive period $(a_j - a_i) = A$. If such continuous contact occurs in the Oth generation over a range nA, where $a_j/A > n > 1$, then a simple reflection shows that in the next generation (the first) contact will occur over a range $(n - 1)A$, in the second generation over $(n - 2)A$, etc. A time is therefore reached (in practice very soon), when contact is over a period less than the period of reproduction A. After that the argument set forth above applies.

If the curve representing the original age distribution contacts with one of the tangent curves all the way from $a = 0$ to $a = a_j$, so that $a_j/A \leqq n$, then, of course, the fixed age distribution is practically established *ab initio*, or at any rate from the moment the original population above reproducing age a_j has died out.

It remains to consider the effect of variability in the form of $\beta(a)$ with changes in $c(t,a)$. Some such variability undoubtedly exists owing to the influence of the ages of the male and female constituents of the population upon the frequency of matings. We may, nevertheless, in this case also, define a minor and a major tangent curve (5), (6) in terms of the value of r given by equation (4); in order, however, to make this value determinate, it is now necessary to make some definite disposition regarding the form of $\beta(a)$, which is now variable. Merely for purposes of defining r, we shall suppose that the function $\beta(a)$ under the integral sign has that particular form which corresponds to the fixed age distribution.[5]

We cannot, however, now reason, as before, that the portion of the population represented by the shaded area in fig. 1 will, by itself, reproduce its own form of age distribution. For its constitution as to sex will in general differ from that of the "normal" population with self-perpetuating age distribution. We must therefore consider three different possibilities:

1. The shaded area alone will produce a population exceeding at all ages the normal or fixed type continuation (8) of the original shaded area. Should this be the case, then the argument presented with regard to the case of invariable $\beta(a)$ holds *a fortiori*, so far as the minor tangent curve is concerned.

2. The shaded area alone will produce a population deficient at some or all ages, as compared with the normal type continuation (8) of the original shaded area. In that case two alternatives present themselves:

a. The deficiency is more than counterbalanced by the additional population produced by that portion of the original population which is represented by the dotted area in fig. 1. In this case also the original argument, so far as the minor tangent curve is concerned, applies essentially as before.

b. After the contributions from all parts of the population have been taken into account, there remains an unbalanced deficiency short of the population defined by (8), in the population resulting from that originally present. In such case the argument presented on the assumption of invariable $\beta(a)$ fails, and the population may move away from, not towards the age distribution(3). Stability of the fixed age distribution may not extend to such displacements as this.

Similar reflections apply, *mutatis mutandis*, as regards the major tangent curve.

If conditions (1) or (2) prevail with respect to the minor, and corresponding conditions with respect to the major tangent curve, then we can argue as in the case of invariable $\beta(a)$, that after expiration of the existing generation a new pair of tangent curves can be drawn, which will lie within the pair defined by (8), (9). And if conditions (1) or (2) still persist with reference to the new tangent curves, the same process of closing in these curves at the expiration of the current generation can be repeated, and so on, as in the argument first presented. Now conditions (1) or (2) will thus continue to prevail for each new pair of tangents, if, as the minor and major tangents close up, $\beta(a)$ approaches a limiting form. In such case, therefore, the age distribution defined by (3), (4) is stable even for displacements of any magnitude, provided always that conditions (1) or (2) prevail, as indicated.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 71.

[2] These circumstances are: (*a*) an invariable life curve (life table); (*b*) an invariable ratio of male to female births; (*c*) an invariable rate per head of procreation at each year of age, for any given sex-and-age distribution in the population.

[3] Lotka, *Amer. J. Sci.;* **26,** 1907 (21); *Ibid.,* **24** (199); *J. Wash. Acad. Sci.,* **3,** 1913 (241, 289).

[4] Sharpe, F. R., and Lotka, A. J., *Phil. Mag.,* April 1911, 435.

[5] A certain ambiguity is introduced here by the fact that, with $\beta(a)$ of variable form, equation (4) might have more than one real root for r. In practice, however, in a human population at any rate, probably only one such root exists, or has any effect upon the course of events.

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PROJECTIVE AND AFFINE GEOMETRY OF PATHS

By O. Veblen

Department of Mathematics, Princeton University

Communicated, October 31, 1922

1. A path is defined as any curve given parametrically by a set of solutions of the differential equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \quad \Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha} \tag{1.1}$$

The same system of paths may, however, be defined in terms of the same system of coördinates by another set of differential equations

$$\frac{d^2x^i}{dt^2} + \Lambda^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0, \quad \Lambda^i_{\alpha\beta} = \Lambda^i_{\beta\alpha} \tag{1.2}$$

The functions $\Gamma^i_{\alpha\beta}$ give rise to one definition of covariant differentiation and infinitesimal parallelism, and the functions $\Lambda^i_{\alpha\beta}$ to a different definition of these operations and relations. But both types of parallelism and of covariant differentiation refer to the same system of paths.

The theorems which state general properties of a system of paths, without restriction as to the scope of the theorems, constitute a projective geometry of paths. The theorems which state properties of the paths and of a particular set of functions, $\Gamma^i_{\alpha\beta}$, i.e., of a particular definition of infinitesimal parallelism, constitute an affine geometry of paths.

The discovery that there can be more than one affine geometry for a given system of paths is due to H. Weyl who also obtained an important tensor which he calls the *projective curvature.* Weyl's results are published in the *Göttinger Nachricten* for 1921 (p. 99), which has only recently reached this country.

The question whether there could be two sets of differential equations such as (1.1) and (1.2) for the same set of paths was raised by Professor Eisenhart (cf. this volume, p. 233) and answered in the negative under the assumption that "the parameter s is the same for all the paths." The following paragraphs will serve to show wherein this assumption restricts the problem and also provide a second method of approaching the projective geometry of paths.

2. Let us first see how the differential equations (1.1) and (1.2) can give rise to the same system of paths. Any solution of the differential equations (1.1) can be written in the form (cf. p. 193 of this volume)

$$x^i = q^i + \psi^i(\xi^1 s, \xi^2 s, \ldots, \xi^n s) \tag{2.1}$$

in which $q^1, q^2, \ldots, q^n$ are the coördinates of an arbitrary point and $\xi^1, \xi^2, \ldots, \xi^n$ an arbitrary set of values of $dx^1/ds, dx^2/ds, \ldots, dx^n/ds$ at this point. The solution of (1.2) representing the same paths as (2.1) may be written in the form

$$x^i = q^i + X^i(\eta^1 t, \eta^2 t, \ldots, \eta^n t) \tag{2.2}$$

in which $\eta^1, \eta^2, \ldots, \eta^n$ is a set of values of $dx^1/dt, dx^2/dt, \ldots, dx^n/dt$ at the point q. By setting up a correspondence between the values of s and t which correspond to the same point of this path we define s as a function of t. This function depends on the point q and the direction of the path through q. Hence if $y^i = \eta^i t$, we may write

$$s = f(q^1, q^2, \ldots, q^n, y^1, y^2, \ldots, y^n) \tag{2.3}$$

for the value of s which corresponds to the same point as $q^1, q^2, \ldots, q^n$ and $y^1, y^2, \ldots, y^n$. As a transformation of the differential equation this may also be written.

$$s = f\left(x^1, x^2, \ldots, x^n, \frac{dx^1}{dt}t, \frac{dx^2}{dx}t, \ldots, \frac{dx^n}{dt}t\right) \tag{2.4}$$

Let us now multilpy (1.1) by $(\partial f/\partial t)^2$ and add a term to each side so as to obtain

$$\frac{d^2x^i}{ds^2}\left(\frac{\partial f}{\partial t}\right)^2 + \frac{dx^i}{ds}\frac{\partial^2 f}{\partial t^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\left(\frac{\partial f}{\partial t}\right)^2 = \frac{dx^i}{ds}\frac{\partial^2 f}{\partial t^2} \tag{2.5}$$

which is the same as

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \frac{dx^i}{dt}\Theta\left(x^1, x^2, \ldots, x^n, \frac{dx^1}{dt}, \frac{dx^2}{dt}, \ldots, \frac{dx^n}{dt}, t\right) \tag{2.6}$$

a differential equation in which t has exactly the same significance as in (1.2).

3. Let us now subtract (1.2) from (2.6). The result is

$$\varphi^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \Theta\frac{dx^i}{dt} \tag{3.1}$$

in which we are defining the functions $\varphi^{\,i}_{\alpha\beta}$ by the equations

$$\varphi^{\,i}_{\alpha\beta} = \Gamma^{\,i}_{\alpha\beta} - \Lambda^{\,i}_{\alpha\beta} \tag{3.2}$$

The functions thus defined are the components of a tensor which is contravariant with respect to i and covariant with respect to α and β. This is because a transformation to new independent variables $z^1, z^2, \ldots, z^n$, changes the Γ's according to the formula

$$\Gamma^{\,i}_{\alpha\beta}(z) = \Gamma^{\,j}_{\gamma\delta}(x)\ \frac{\partial x^\gamma}{\partial z^\alpha}\,\frac{\partial x^\delta}{\partial x^\beta}\,\frac{\partial z^i}{\partial x^j} + \frac{\partial^2 x^j}{\partial z^\alpha \partial z^\beta}\,\frac{\partial z^i}{\partial x^j} \tag{3.3}$$

This tensor defines a covariant vector by means of the formula

$$\varphi_\alpha = \frac{1}{n+1}\,\varphi^{\,i}_{i\alpha} \tag{3.4}$$

By eliminating Θ from the equations (3.1) we obtain

$$\left(\varphi^{\,i}_{\alpha\beta}\ \frac{dx^i}{dt} - \varphi^{\,j}_{\alpha\beta}\ \frac{dx^i}{dt}\right)\frac{dx^\alpha}{dt}\,\frac{dx^\beta}{dt} = 0 \tag{3.5}$$

Remembering that the values of $dx^1/dt, dx^2/dt, \ldots, dx^n/dt$ are arbitrary it is easy to infer from (3.5) that

$$\varphi^{\,i}_{\alpha\beta} = \delta^{\,i}_{\alpha}\,\varphi_\beta + \delta^{\,i}_{\beta}\,\varphi_\alpha \tag{3.6}$$

where δ^i_α is 1 or 0 according as $i = \alpha$ or $i \neq \alpha$.

When (3.6) is substituted in (3.1) there results

$$2\,\varphi_\gamma\,\frac{dx^\gamma}{dt} = \Theta. \tag{3.7}$$

4. Conversely it can be shown that if we start with the differential equation (1.1) and any covariant vector φ_α, we can find a second differential equation of the type (1.2) which defines the same set of paths as (1.1). The first step is to define the functions $\varphi^{\,i}_{\alpha\beta}$ by means of equations (3.6). Since δ^i_α is a tensor and φ_α is a covariant vector it follows that $\varphi^{\,i}_{\alpha\beta}$ is a tensor of the third order. We then define $\Lambda^{\,i}_{\alpha\beta}$ by means of the equations (3.2) and write the differential equations (1.2) which are by definition identical with

$$\frac{d^2x^i}{dt^2} + \Gamma^{\,i}_{\alpha\beta}\,\frac{dx^\alpha}{dt}\,\frac{dx^\beta}{dt} = 2\,\varphi_\alpha\,\frac{dx^\alpha}{dt}\,\frac{dx^i}{dt} \tag{4.1}$$

If we compare (4.1) with (2.5) we see that (4.1) is obtainable from (1. 1) by means of the substitution

$$s = e^{2\varphi_\gamma \frac{dx^\gamma}{dt}} + h(x) \tag{4.2}$$

where $h\,(x)$ is an arbitrary function of $x^1, x^2, \ldots, x^n$.

It has been proved that: (1) If (1.1) and (1.2) are differential equations which define the same system of paths then the functions $\Gamma^i_{\alpha\beta}$ and $\Lambda^i_{\alpha\beta}$ are related by the equations

$$\Gamma^i_{\alpha\beta} - \Lambda^i_{\alpha\beta} = \varphi^i_{\alpha\beta} = \delta^i_{\alpha}\varphi_{\beta} + \delta^i_{\beta}\varphi_{\alpha} \tag{4.3}$$

in which φ_α is a vector. (2) If (1.1) are the differential equations of a system of paths and φ_α is any covariant vector then there exists a second set of differential equations (1.2) for the same system of paths, the functions $\Lambda^i_{\alpha\beta}$ being defined by (4.3).

The formula (3.7) reduces to that obtained by Eisenhart on page 235 of this volume in the case when s is a function of t alone. This makes it evident that his work must be taken as referring to a geometry in which there is an element of distance ds defined by means of functions of position. In this sense his theorem on page 235 will be found not to be in contradiction with the theorem (2) above.

A DIRECT METHOD OF TESTING COLOR VISION IN LOWER ANIMALS

By W. F. Hamilton

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1. When spectral lights of different wave-lengths are directed on the photoreceptors of lower organisms, their reactions show that one of the two lights is, physiologically, the more effective. Even when the unequal distribution of energy in the spectrum is equated for, there still remains a difference in the physiological effectiveness of the stimuli. (Laurens and Hooker.[1]) This difference must be dependent upon wave-length. But it is also evident to the color blind eye (Koenig[2]). We cannot therefore argue that the differential response on the part of the organism is proof of color division.

Researches which have approached the question of color vision in lower organisms by means of *conditioned behavior* have not been done with pure spectral lights. Even when pure spectral lights are used, the difference in brightness must be equalized, not, however, in accordance with the luminosity curve of the human eye, but with that of the organisms whose color vision is to be tested. Obviously it is difficult to obtain a luminosity curve free from extraneous factors from the reactions of organisms sufficiently complex and intelligent to exhibit conditioned behavior.

The method of conditioned behavior is not applicable to lower organisms and it meets with difficulties when applied to higher animals. Therefore, if we would study the evolution of reactions to color, we must cast about for another method.

Most theories of color vision postulate receptive systems maximally sensitive to certain wave-lengths which differ in some manner from those which are maximally sensitive to other wave-lengths. The difference, according to some theories, lies in the receptive substance of the eye (Young-Helmholtz, Hering); according to others, there are different "color perceiving" mechanisms in the brain which serve for perception of the different fundamental colors. (Edridge Green,[3] Houstoun,[4] and the various "Zone Theories.") Work on the human eye (Burch,[5] Abney,[6] Frank Allen[7] and others) shows that these receptive systems can be separately fatigued by strong monochromatic light.

If we can get evidence that the photo receptors of lower organisms can be differentially fatigued in a similar manner, it would indicate that the receptive system which is maximally sensitive to certain wave-lengths differs in some respect from the receptive system which is maximally sensitive to other wave-lengths, and this could be construed into something analogous to color vision. The following is a preliminary report of the results of experiments recently performed with this in view.

2. The animal decided upon was the fruit-fly, Drosophila. Specimens were kept in the dark over night and used in the dark adapted state. The apparatus consisted of two Hilger wave-length spectrometers** used as monochromatic illuminators and so set as to throw beams of light into opposite ends of a small horizontal tubular glass cell containing the flies. The sources of light were two 500-watt Mazda C stereoptican lamps.

The intensities of the light beams, differing as to wave-length were so regulated that the animals were not oriented in either direction. The lights were physiologically equal. As soon as this was accomplished one beam of light, called for convenience the "stimulating light," was intercepted by a screen and the animals subjected to the action of the other ("fatiguing") light. As soon as a marked decrease in the positive reaction of the flies to the "fatiguing" light was observed, the screen was removed and the animals subjected to the "stimulating" light again.

As soon as this was done, the animals usually oriented and crawled toward the "stimulating" light in spite of the continued action of the "fatiguing" light. The operation of the "fatiguing" light had thus upset the physiological balance between the two lights by fatiguing the apparatus maximally sensitive to this light, more than the apparatus maximally sensitive to the "stimulating" light. When such a test had been complete, it was repeated immediately, with the light that had been used as the "fatiguing" light arranged to "stimulate" and vice versa. In the second test, the animals were, of course, not dark adapted.

If, however, the two lights were of the same or nearly the same wave-lengths the results were quite different. The lights were balanced, the "stimulating" light intercepted, and the animals subjected to the "fatiguing" light. There was evidence of fatigue, but when the "stimulating" light was again allowed to fall on the flies, it was easily seen that the fatigue was not differential. The animals distributed themselves at random, between the two ends, just as they did when the lights were first balanced.

The spectrum was then scaled. The smallest difference in wave-length which showed conclusive evidence of differential fatigue was the unit used. Beginning at 385 $m\mu$, which is as low as the calibration of the instruments extend, the values were found as follows:

385–410, 410–430, 430–450, 450–500 $m\mu$

The last determination was rather doubtful. At the intensity available the orienting effect of light above 500 $m\mu$ was insufficient to give conclusive evidence of differential fatigue. The distribution of these points and the fact that "hue perception" is at its maximum between 410 $m\mu$ and 450 $m\mu$ makes it appear that there may be two receptor systems, one for the blue-violet and one for the blue-green. Evidence for a third component seems to be lacking. According to Steindler[8] the hue perception curve of the normal human eye indicates the existence of three receptor systems.

3. It is planned to extend this work to other organisms and to work out the energy—time relationships of differential fatigue with different pairs of wave-lengths at different intensities.

The method should prove of value in analyzing the wave-length receptors of lower organisms, and its modifications and applications are numerous. It is applicable to the investigation of the pupil reactions, of the action current of the optic nerve and to similar problems. Perhaps one of its most interesting applications would be to ascertain the course of evolution of the color sense among lower organisms by means of photic orientation in crossed beams of light. The chief value of the method is its objectiveness.

[1] Laurens, H., and Hooker, H. O., 1920, Studies on the relative physiological value of spectral lights. *J. Exper. Zoöl.*, **30,** 345–68.

[2] Koenig, *Ges. Abhandlungen*, Berlin.

[3] Edrige Green, F. W., 1920, *The Physiology of Vision.* London.

[4] Houstoun, R. A., 1916, A theory of color vision. *Proc. Roy. Soc. London,* **A92,** 424–32.

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[7] Allen, Frank, 1920, Persistence of vision and the primary color sensations. *Amer. J. Physiol. Optics*, **1**, 94–134.

[8] Steindler, 1906, Der Farbenempfindlichkeit des normalen und farbenblinden Auges. *Sitz. Weiner Akad.*, **105**, 11a, 115–6.

** One of these instruments belongs to this Laboratory, while the other was loaned to us by the kindness of Professor G. H. Parker, Harvard University.

CHARACTERISTICS OF A SHORT WAVE OSCILLATOR AT VERY LOW PRESSURES

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Various investigators have experimented with three element vacuum tubes oscillating at a wave-length of the order of one meter. All of them have used commerical tubes, having a cylindrical arrangement of filament, grid, and plate.

Barkhausen and Kurz[1] discovered a type of oscillation which was apparently due to the motion of electrons in the tube itself, and was independent of the inductance and capacity of the external circuit. The computed time for an electron to travel across the tube, from filament to plate and back, under the potentials applied, agreed fairly well with the period of the waves as measured by means of Lecher wires coupled to the tube.

Whiddington[2] experimented with a similar type of oscillation, but he observed much lower frequencies and attributed the observed effects to the motion of ions instead of electrons. An explanation was given which involved a discontinuous emission from the filament.

Gill and Morrell[3] have experimented extensively with the Barkhausen type of oscillation, using commercial tubes made by the Marconi company, and have given an explanation of the effect which involves a natural mode of oscillation of the electrical system connected to the tube.

The work of Barkhausen and Kurz and that of Gill and Morrell was done with so-called "hard" tubes, which in the process of manufacture have been thoroughly heated to remove occluded gas and then sealed off from the vacuum pump. These men have assumed that they were dealing with a purely electronic phenomenon, and that no gaseous ionization was concerned.

The essential departure in the experimental arrangement in the present investigation was the use of a tube which was left permanently connected to the vacuum pumps, and was made with a ground joint to make the inside parts accessible for modification, replacement of burned out filaments, etc. Also, more power was used than by other investigators, electron currents as high as 300 milamperes and at voltages up to 700 being used at times.

The electrical connections were made as indicated in figure 1, except that the filament heating circuit is not shown. The Lecher wires, LL, were closely coupled to, but not in metallic contact with, the grid and plate leads, which were brought straight in through the sides of the tube. A sliding bridge, B, across the Lecher wires, carried a crossed wire thermocouple which was connected to a sensitive galvanometer by long, twisted leads. This thermocouple served to measure the strength of oscillations, and by sliding the bridge along the wires, the points of maximum deflection gave the loops of the standing waves, and the distance between these points gave the half wave-length of the oscillations. In the discussion which follows, by strength of oscillations is meant the deflection of the thermocouple galvanometer when the bridge position is adjusted so that this deflection is a maximum.

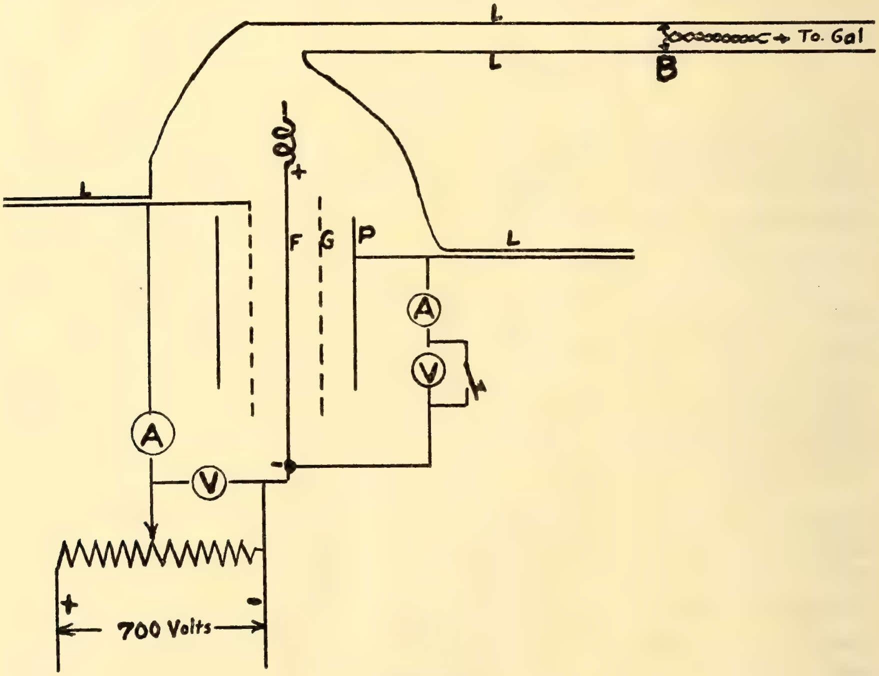

FIGURE 1

Oscillations of the Barkhausen type, at wave-lengths of from 50 to 200 cm., were obtained. The unusual characteristic of a tube when oscillating in this way is that there is a negative current to the plate, which may flow against a negative potential of 100 volts or more with respect to the filament, the grid being at a positive potential of several hundred volts. If the plate is simply connected to the filament through a voltmeter, this voltmeter may read as high as 150 volts, or if connected through an ammeter, the negative current to the plate may be as much as 25% of the total electron current from the filament. The magnitude of this negative current or voltage varies greatly with the temperature of the filament, grid voltage, and gas pressure. There is never oscillation in the Lecher wires without a negative current or potential on the plate, but this negative current may be present and yet the Lecher wires show very feeble or no oscillations. The characteristics of this negative current and the associated oscillations have been studied as affected by changes of grid voltage, of electron current, and of gas pressure.

It was found, very unexpectedly, that both the negative plate current and any sign of oscillation, as indicated by the Lecher wires, ceased abruptly

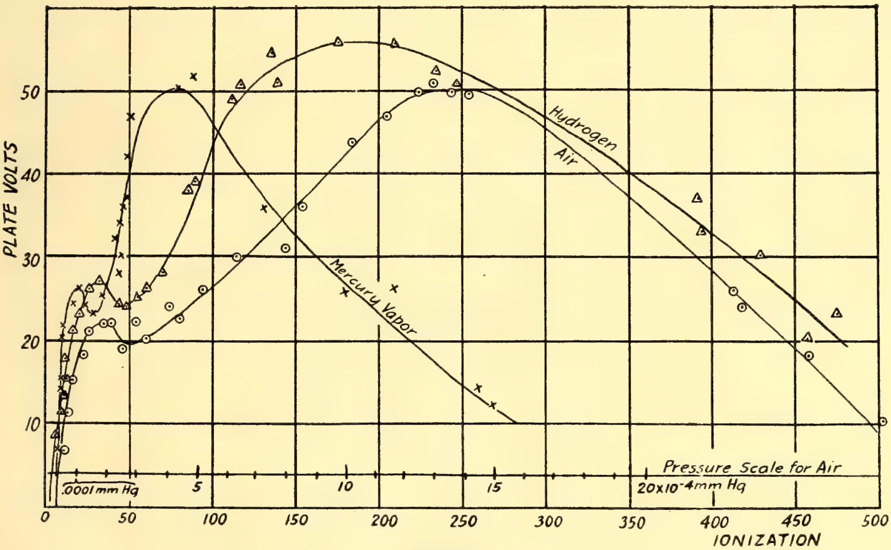

FIGURE 2

at very low pressures. This is shown in the curves of figure 2 for plate voltages, and in figure 3 for the strength of oscillation in the Lecher wires. These observations were all made with the grid at 250 volts and the filament heating currents such that the electron current was 200 milliamperes. The pressure of the residual gas in the tube was measured with an ionization manometer. The pressures of hydrogen and air were controlled by adjusting the flame under the mercury vapor pump, with liquid air on the trap to keep out mercury vapor. The mercury vapor pressure was controlled by keeping the liquid air trap (with a pool of mercury at the bottom) at different temperatures, the mercury vapor pump operating at its maximum rate to reduce the pressure of any gas to the minimum. The curves are plotted with the readings of the ionization manometer as abscissae. This was done because it is probable that the essential thing is not the presence of the gas itself, but of gaseous ions. Since different gases at the same pressure give different amounts of ionization when under the same ionizing agent, it seemed better to plot their ionization instead of pressure. The pressure corresponding to a given ionization current is for mercury about half, and for hydrogen about twice that for air. The pressure scale for air is given on the curves, so by multiplying and dividing by 2 the approximate pressures of hydrogen and mercury, respectively, will be obtained.

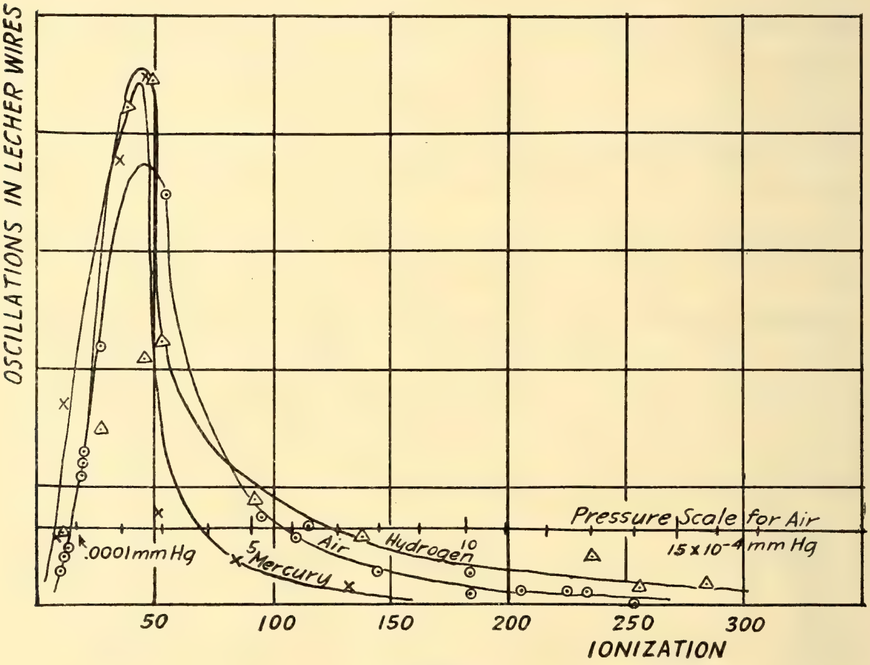

FIGURE 3

These curves apparently indicate that a small amount of ionization is necessary for this type of oscillation. The kind of gas does not seem important. Since the gas pressure at which the oscillation stops is so low, being less than .00005 mm. there can be no connection with the mean free path of the ions, for at these pressures the path is many times longer than the dimensions of the tube. It is not likely that the commercial tubes used by other investigators are sufficiently exhausted to give any indication of the dropping off of the curves at the low pressure side. So far, no staisfactory explanation has been found for the profound effect of such a small amount of gas or the behavior of the tube.

The investigation is being continued and a more detailed account will be published in the near future.

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[2] Whiddington, *Radio Review,* Nov. 1919, p. 53.

[3] Gill and Morrell, *Phil. Mag.,* **44,** No. 259, July 1922, p. 161.

THE REFRACTION OF X-RAYS IN CALCITE

By Bergen Davis and H. M. Terrill

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The effect of refraction in X-ray spectra has been discussed by Stenström,[1] who made some determinations of the index of refraction from the relative displacement of the several orders. In most cases the effect was too small to admit of measurement, but for sugar and gypsum crystals he obtained some values for wave-lengths greater than 2.5 Å.

The present paper applies a modification of the same method to the reflection from calcite of the $K\alpha_1$ line of Molybedenum, 70783 Å.

Since we measure the angle from the crystal face, and not the normal, the customary equation for the index of refraction becomes

$$\nu = \frac{\cos\theta}{\cos\theta'}, \qquad (1)$$

where θ is the glancing angle outside the crystal and θ' the angle of the beam inside. We will use a subscript to indicate orders higher than the first. Placing $\nu = 1 - \delta$, Stenström computes the values of δ from the equation

$$\delta = \frac{\left(\frac{\sin\theta_m}{m}\right)^2 - \left(\frac{\sin\theta_n}{n}\right)^2}{2\left(\frac{\cos\theta_m}{m}\right)^2 - 2\left(\frac{\cos\theta_n}{n}\right)}$$

where m and n are any two orders.

As the shift is small in any case, it seemed desirable to express the value of δ directly in terms of angle, in order to more readily determine the effect of errors of observation which are liable to be of the same order of magnitude as δ itself. Now from (1) we have

$$\sin^2\theta' = 1 - \nu^{-2}\cos^2\theta$$

Since δ is small, we write

$$\nu^{-2} = 1 + 2\,\delta$$

whence

$$\sin\theta = [1 - \cos^2\theta\,(1 + 2\,\delta)]^{1/2} = [\sin^2\theta - 2\,\delta\cos^2\theta]^{1/2}$$

and again neglecting terms containing higher powers of δ.

$$\sin\theta' = \theta - \delta\,\frac{\cos^2\theta}{\sin\theta} \tag{2}$$

Taking arc sin of each side, we have since δ is small,

$$\theta' = \theta - \delta\,\frac{\cos^2\theta}{\sin\theta}\cdot\frac{1}{\cos\theta} \tag{3}$$

$$\theta' = \theta - \delta\cot\theta$$

Now in this equation, θ is the crystal angle measured, hence by determining the angles of any characteristic line for several orders, it will be possible to determine δ, or at least to find the limits of its magnitude. The angles being measured, those of higher orders are reduced to the equivalent first order angle, so that all angles may be compared directly. The proportion that each is affected by refraction being known from (3), the absolute value of δ is readily determined.

Instead of using (3), it is possible to substitute (2) in the relation

$$n\,\lambda = 2\,d\sin\theta_n'$$

obtaining, since δ is small

$$\frac{n\,\lambda}{2\sin\theta_n} = d\,(1 - \delta\cot^2\theta_n) \tag{4}$$

thus determining δ from the increase in apparent grating space with order. However, since the limits of error have to be carried through in the computations, the previous method seems to be more convenient.

The spectrometer used has been described elsewhere.[2] The tube was of the water cooled type with a molybdenum target. Measurements of θ were made for the first three orders of the $K\alpha_1$ line, using a crystal of clear Iceland spar. The measurements were gone over several times and it was found that the angles could be repeated to within 15″. This is just about the limit of accuracy of the instrument which was determined by other methods to be about 20″.

The following mean values were obtained.

$$\theta = 6^\circ\ 42'\ 43'' \pm 10''$$
$$\theta_2 = 13^\circ\ 30'\ 45'' \pm 10''$$
$$\theta_3 = 20^\circ\ 31'\ 22'' \pm 10''$$

where the additional terms represent the limits of error.

It should be remarked that if the crystal is not mounted so that the mean reflecting plane is directly over the center of rotation, and the face of the crystal be slightly curved, a shift of the lines will be obtained, that might be taken for refraction. To eliminate this possibility, the crystal was mounted on a slide with a micrometer screw attached, so that it might be moved in a direction perpendicular to its face. The crystal being mounted as nearly as possible over the center of rotation by mechanical methods, the adjustment was changed by small fractions of a millimeter, reading each time on both sides the crystal and chamber angles of the α_1 line. If curvature is present, the crystal angle will change as the crystal is moved. Twice the crystal angle is plotted against the readings of the micrometer head and on the same sheet, the chamber angle is also plotted against these readings. The intersection of the two lines gives the proper micrometer setting, since the crystal must be over the center, when the chamber angle is exactly twice the crystal angle. With the crystal used, it was found that the curvature was very slight, a shift of less than one second of arc being found within the range of adjustment. This was checked for each order.

We proceed then to reduce the angles measured to the corresponding first order angles. Now $\theta_n = \theta_n' + \Delta$ where Δ includes the slight increase due to refraction and also the error of observation. Since Δ is small

$$\sin\theta_n = \sin\theta_n' + \Delta\cos\theta_n$$

$$\frac{\sin\theta_n}{n} = \sin\theta_n' + \Delta\frac{\cos\theta_n}{n}$$

$$\operatorname{arc\,sin}\left(\frac{\sin\theta_n}{n}\right) = \theta' + \Delta\frac{\cos\theta_n}{n\cos\theta}$$

$$\theta' = \operatorname{arc\,sin}\left(\frac{\sin\theta_n}{n}\right) - \Delta\frac{\cos\theta_n}{n\cos\theta}$$

and the coefficient of Δ can be computed for each order, it being .49 for the second and .31 for the third. Multiplying the errror limits by these coefficients and introducing the refraction from (3) which is also multiplied by them, we have the following values for θ':

$$\begin{aligned}\theta' &= 6^\circ\ 42'\ 43'' \pm 10'' - 1''.76\cdot\delta\cdot10^6\\ &= 6^\circ\ 42'\ 33'' \pm 5'' - 0''.63\cdot\delta\cdot10^6\\ &= 6^\circ\ 42'\ 38''.{}^5 \pm 3'' - 0''.27\cdot\delta\cdot10^6\end{aligned}$$

and we have to determine δ from these equations.

Plotting the values of the angles against the order, using lines covering the limits of error instead of points to locate the extremities of the ordinates, we have to pass through these lines a curve whose ordinates are proportional to the coefficients of δ. Now the values found do not lie well on any curve of this type, the second order being entirely too low. The curves that fitted most nearly gave a value of $\delta = 3 \times 10^{-6}$ while the error limits would permit of values ranging from zero to several times this. It will be seen that this corresponds to a shift of the first order of 5″, so that for this wave-length, the effect of refraction is very slight.

This value $\delta = 3 \times 10^{-6}$ necessarily contains large possible errors. It is, however, of the same order of magnitude as that obtained by the method of total reflection.

An independent determination of δ was obtained by measuring the angle of total reflection as suggested by the experiments of A. H. Compton[3] on total reflection from glass and lead. The ionization chamber was shielded from the direct beam and the spectrum searched at angles close to grazing incidence. Total reflection appeared as a line, approximately 4′ wide at half maximum, slightly unsymmetrical and of an intensity about 25 times the surrounding values which coincided with the leak. To test if this line consisted principally of the monochromatic radiation corresponding to the $K\alpha$ line, a zirconium screen was introduced. This diminished the intensity without materially changing the shape of the line or shifting its position.

Measurements were made on each side of the zero position, and the mean of the readings gave 6′ 30″, the value being good to 30″. Now for total reflection, putting $\theta' = 0$ in (1) we have

$$\nu = \cos\theta \text{ or } \delta = \frac{\sin^2\theta}{2}$$

whence

$$\delta = 1.7 \pm .5 \times 10^{-6}$$

Comparing this with the theoretical value from the Lorentz equation

$$\delta = \frac{n\,e^2}{2\,\pi\,m}\cdot\frac{1}{\nu}$$

we take the density of calcite as 2.71 giving 81.5×10^{22} electrons per cubic centimeter, and place ν equal to the frequency of the molybdenum $K\alpha$ line, whence $\delta = 1.85 \times 10^{-6}$.

Determinations of the total reflection made with a second crystal gave results which checked with the preceding. Incidently it developed that the face of this second crystal had been ground at a slight angle to the reflecting planes, the crystal zero determined by the angle of total reflection differing from that obtained from the line spectra by an angle of 44′. This suggests an extension of the method of determining the refraction from the shift of the spectral lines, for if the face of the crystal were ground at the proper angle to the reflecting planes, the shift would be increased enough to make accurate determinations possible.

[1] *Experimentelle Untersuchungen der Röntgenspektra,* Lund 1919.

[2] *J. Amer. Opt. Soc.,* May 1922.

[3] *Physic Rev., Ithaca,* July 1922.

THE COMPRESSIBILITY OF METALS AT HIGH PRESSURES

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This note briefly summarizes results which will be found in full detail in a forthcoming number of the PROCEEDINGS of the American Academy of Arts and Sciences. A considerable part of the expenses of the experiment was defrayed by a generous appropriation from the Rumford Fund of the American Academy.

A new method has been devised by which the difference of the linear compressibility of a solid in the form of a rod or wire and the linear compressibility of iron may be measured with an accuracy high enough to give the variation of compressibility with pressure or temperature. In addition, I have improved the method which I previously used[1] for the measurement of the absolute linear compressibility of iron, so that I have now measured the change of linear compressibility with pressure with some accuracy, and I have also obtained improved values for the change of linear compressibility with temperature. By combining these absolute measurements on iron with the measurements of the difference of compressibility the absolute linear compressibility of a number of metals has been obtained.

If the metal is equally compressible in all directions, the absolute volume compressibility may be calculated from the linear compressibility. The great majority of metals crystallize in the cubic system or in the hexagonal close packed arrangement of spheres, and for these the compressibility is equal in all directions. Table I gives the compressibility, χ (defined as $(\partial v/\partial p)_T$ where v is the volume of the amount of the metal that under standard conditions occupies 1 cc.) at atmospheric pressure at 30°, the pressure derivative of the compressibility, and the change of compressibility per degree rise of temperature for a number of such metals. The unit of pressure is the kilogram per square centimeter, the pressure range of the experiments was 12000 kg./cm.[2], and measure-

TABLE I

SUBSTANCE	$-\chi$ AT 30° $\times 10^7$	$-\frac{1}{\chi}\left(\frac{\partial\chi}{\partial p}\right)_\tau \times 10^5$	$+\frac{1}{\chi}\left(\frac{\partial\chi}{\partial\tau}\right)_p \times 10^4$	SUBSTANCE	$-\chi$ AT 30° $\times 10^7$	$-\frac{1}{\chi}\left(\frac{\partial\chi}{\partial p}\right)_\tau \times 10^5$	$+\frac{1}{\chi}\left(\frac{\partial\chi}{\partial\tau}\right)_p \times 10^4$
Al	13.43	0.74	5.4	Ni	5.29	0.802	2.5
Ca	56.97	1.66	6.0	Pd	5.28	0.80	1.3
Ce	35.74	1.06	0.4	Pt	3.60	1.00	3.7
Co	5.39	0.78	3.0	K	356.5	16.1	35.1
Cu	7.19	0.72	4.6	Ag	9.87	0.90	3.8
Ge	13.78	0.98	—2.3	Na	156.2	3.8	22.
Au	5.77	1.08		Sr	81.87	1.78	2.2
Fe	5.87	0.72	2.3	Ta	4.79	0.10	1.0
Pb	23.73	1.46	5.6	W	2.93	1.02	1.5
Li	86.92	2.24	7.2	Tl*	34.2	3.36	12.2
Mg	29.60	1.36	2.8	Ur†	9.66	0.52	—2.5
Mo	3.47	0.72	0.6				

* The crystal system of Tl has not yet been determined. The compressibility listed above was determined from a single specimen and is very materially higher than the value of other observers. It is therefore probable that the crystal system of Tl is not cubic.

† Crystal system of Ur not yet determined.

ments were made at two temperatures, 30° and 75°. The relation between volume and pressure of all these metals, except Na, K, and Tl, could be represented within the accuracy of the experiments by a second degree equation in the pressure. For the three exceptional metals, however, the relation is more complicated, and the coefficients listed in table I are only the initial values at atmospheric pressure. Of the above list of metals, the compressibility of Ge, Sr, and Ur seems never to have been measured, even at low pressures.

It is to be noticed that the order of magnitude of the relative change of compressibility with pressure or temperature is the same for all the metals. The compressibility of every metal decreases numerically with rising pressure, and in nearly every case increases with rising temperature. There are two apparent exceptions to this last statement in the cases of Ge and Ur. It is possible that the result found for Ur may be due to experimental error, but this explanation seems less likely in the case of Ge. This metal is known to be abnormal in other respects.

If the metal does not crystallize in a system whose compressibility is the same in all directions, then a complete description of the distortion under pressure requires that measurements be made of the linear compressibility in a number of directions of orientations known with respect to the crystal structure. This elaborate program I have not as yet been able to carry through, but I have been able to establish in several cases that there are very considerable differences of compressibility in different directions. These differences are shown by the fact that measurements on different specimens of the same metal do not yield consistent results, unlike measurements on different specimens of cubic metals. If a non-cubic metal is measured first as a casting and then as an extruded rod, different results will nearly always be found. Results on two different castings, even, are not the same, unless the method of cooling the castings has been the same, so as to ensure the same average orientation of the crystal grains. By measuring the linear compressibility of non-cubic metals prepared in different ways, it is possible to set a lower limit to the differences of compressibility that would be found in different directions in a single crystal. This was the method adopted, and in table II are the results for several non-cubic metals. The values tabulated are the linear compressibility, *k* (defined as $(\partial l/\partial p)_\tau$, where *l* is a length parallel to one of the sides of a body of metal which under standard conditions is a cube one centimeter on a side) at 30° at atmospheric pressure, and its pressure and temperature derivatives.

TABLE II

SUBSTANCE		$-k$ AT 30° $\times 10^7$	$-\frac{1}{k}\left(\frac{\partial k}{\partial p}\right)_\tau$ $\times 10^5$	$+\frac{1}{k}\left(\frac{\partial k}{\partial \tau}\right)_p$ $\times 10^4$
Bi	cast	7.67	0.97	0.9
	extruded	11.78	1.82	3.7
Sb	cast	6.80	1.39	1.0
	cast	4.89	0.93	1.7
Cd	cast	4.72	1.15	5.6
	cast	6.51	1.23	7.4
	uni-crystalline	2.86	0.94	
Sn	extruded	6.51	1.12	6.6
	cast	5.67	0.71	4.7
Te	cast	—2.36		
Zn	cast	1.66		
		5.30		
		7.13		

The three measurements on zinc were on pieces cut from the same original single crystal, oriented so as to give three directions mutually at right angles. For zinc, therefore, these numbers give a fair idea of the difference of compressibility in different directions in the same crystal. It is surprising that the difference is so large. The mean of the values listed above for three perpendicular directions is nearly the same as one-third the average cubic compressibility found by other observers, confirming the essential correctness of these measurements.

The negative compressibility of tellurium in a particular direction is perhaps unexpected, and so far as I know this is the first case found of this kind of behavior, although a somewhat analagous instance, the negative Poisson's ratio of pyrites, has been found by Voigt.

A number of theoretical considerations are suggested by these measurements of compressibility. In the first place, and most important, I have shown in the detailed paper by a dimensional argument that the correct order of magnitude for the compressibilities of all these metals may be found by taking over for the metal the same picture of the structure that has given such suggestive results when applied to the salts, namely the picture of the metal as composed of two interpenetrating lattices, one of positively charged atomic ions, and the other of negative electrons, the electron lattice in the case of the metal taking the place of the lattice of negatively charged atoms in the case of the salts, as for example the chlorine lattice in NaCl. The same conclusion as to the structure of a metal has been recently drawn by Kraus[2] on the basis of a large number of chemical facts.

The expression of Born[3] for the compressibility of salts, in which a repulsive potential between atoms as the inverse ninth power is found to give good values for the compressibility, cannot be extended to metals, for neither is the absolute value of the compressibility nor its variation with pressure such as would be given by this sort of a potential. There seems to be no simple repulsive potential that will account for the compressibility data for the various metals. It is possible, however, to represent the repulsive potential by an arbitrary function, and then to find the numerical value of several of its derivatives from experimental data. For those metals which crystallize face-centered cubic, and which therefore probably consist of interpenetrating lattices of ions and electrons of the type of NaCl, it is possible to calculate the grating constants, and to evaluate the first three derivatives of the repulsive potential in terms of the absolute dimensions of the crystal, the compressibility, and the change of compressibility with pressure. These derivatives for a number of metals are listed in the detailed paper. The sign of the derivatives is alternately positive and negative, and their successive absolute values differ by factors of the order of magnitude of the distance of separation of atomic centers. This is all as one would expect. To make more detailed use of these numerical values of the derivatives demands that more attention be paid to the details of the atomic structure than we have hitherto been able to do.

An effect recently discussed by Schottky,[4] namely a distortion of the atoms under pressure, I have shown by a detailed numerical examination to be of such a magnitude as to essentially modify the simple calculations of compressibility given by Born.

By the use of simple thermodynamics I have been able to modify the arguments of Born, which applied only at the absolute zero of temperature, so as to take account of the effect of temperature on the compressibility to be calculated in terms of the lattice structure. The new terms introduced by temperature are not large enough to seriously modify Born's argument.

Finally, with the new experimental data we are now in a position to calculate for the first time the changes in the specific heat of a metal at high pressures. It appears that the specific heat at constant volume decreases under pressure by an amount of the same order of magnitude as the change of volume under the same pressure, but in most cases by a factor several fold greater.

[1] Bridgman, P. W., *Proc. Amer. Acad., Boston,* **44,** 1909 (255–279), and **47,** 1911 (362–368).

[2] Kraus, C. A., *J. Amer. Chem. Soc., Easton, Pa.,* **44,** 1922 (1216–1238).

[3] Born, M., *Verh. Deut. Phys. Ges., Brauschweig,* **21,** 1919 (533–538).

[4] Schottky, W., *Phys. Zs., Leipzig,* **21,** 1920 (232–141).

A FURTHER NOTE ON THE MATHEMATICAL THEORY OF POPULATION GROWTH[1]

By Raymond Pearl and Lowell J. Reed

In an earlier paper in these Proceedings we[2] showed that the expression

$$y = \frac{be^{ax}}{1 + ce^{ax}} \tag{i}$$

gives an excellent fit to the known population growth of the United States since 1790. Since the first paper was published, we have extended and generalized our ideas on population growth with the results herein presented in skeleton outline. A more extended paper, giving a full discussion of our new results and of the pertinent literature is in press in Metron.

Considered generally, the curve

$$y = \frac{b}{e^{-ax} + c}$$

may be written

$$y = \frac{k}{1 + me^{ka'x}}, \tag{ii}$$

where

$$k = b/c,\ m = 1/c,\ \text{and}\ ka' = -a.$$

Now the rate of change of y with respect to x is given by

$$\frac{dy}{dx} = -a'y\,(k-y)$$

or

$$\frac{\frac{dy}{dx}}{y\,(k-y)} = -a'. \qquad \text{(iii)}$$

If y be the variable changing with time x (in our case population) equation (iii) amounts to the assumption that the time rate of change of y varies directly as y and as $(k-y)$, the constant k being the upper limit of growth, or, in other words, the value of the growing variable y at infinite time. Now since the rate of growth of y is dependent upon factors that vary with time, we may generalize (iii) by using $f\ (x)$ in place of $-a'$, $f\ (x)$ being some as yet undefined function of time.

Then

$$\frac{dy}{y\,(k-y)} = f(x)dx.$$

whence

$$\frac{k-y}{my} = e^{-k\int f(x)dx},$$

and

$$y = \frac{k}{1+me^{-k\int f(x)dx}} = \frac{k}{1+me^{F(x)}} \qquad \text{(iv)}$$

where

$$F(x) = -k\int f(x)dx.$$

Then the assumption that the rate of growth of the dependent variable varies as (a) that variable, (b) a constant minus that variable, and (c) an arbitrary function of time, leads to equation (iv), which is of the same form as (i), except that ax has been replaced by $F(x)$. If now we assume that $f(x)$ may be represented by a Taylor series, we have

$$y = \frac{k}{1+me^{a_1x + a_2x^2 + a_3x^3 + \dots\dots\dots + a_nx^n}} \qquad \text{(v)}$$

If

$$a_2 = a_3 = a_4 = a_n = 0$$

then (v) becomes the same as (i).

If m becomes negative the curve becomes discontinuous at finite time. Since this cannot occur in the case of the growth of the organism or of populations, nor indeed so far as we are able to see, for any *phenomenal* changes with time, we shall restrict our further consideration of the equation to positive value only of m. Also since negative values of k would give negative values of y, which in the case of population or individual growth are unthinkable, we shall limit k to positive values.

With these limitations as to the values of m and k we have the following general facts as to the form of (v). y can never be negative, i.e., less than zero, nor greater than k. Thus the complete curve always falls between the x axis and a line parallel to it at a distance k above it. Further we have the following relations:

$$\begin{array}{lll} \text{If} \quad & F(x) \doteq \infty & y=0 \\ & F(x) \doteq -\infty & y=k \\ & F(x) \doteq -0 & y=\dfrac{k}{1+m} \text{ from below} \\ & F(x) \doteq +0 & y=\dfrac{k}{1+m} \text{ from below.} \end{array}$$

The maximum and minimum points of (v) occur where $\dfrac{dy}{dx} = 0$.

But $\dfrac{dy}{dx} = y(k-y) \,.\, F'(x)$,

therefore we have maximum and minimum points wherever $F'(x) = 0$. The fact that $\dfrac{dy}{dx} = 0$ when either $y = 0$ or $y - k = 0$ shows that the curve passes off to infinity asymptotic to the lines $y = 0$ and $y = k$.

The points of inflection of (v) are determined by the intersections of (v) with the curve.

$$y = \frac{k}{2} - \frac{k}{2}\,\frac{F''(x)}{[F'(x)]^2} \tag{vi}$$

Since we are seldom justified in using over five arbitrary constants in any practical problem, we may limit equation (v) still further by stopping at the third power of x. This gives the equation

$$y = \frac{k}{1 + me^{a_1x + a_2x^2 + a_3x^3}}. \tag{vii}$$

If a_n is positive the curve of equation (v) is reversed and becomes asymptotic to a line AB, at $x = -\infty$ and to the x axis at $x = +\infty$. Thus in equation (vii) a_3 negative is a case of growth, and a_3 positive is a case of decay.

Equation (vii) has several special forms that are of interest, among them being a form similar in shape to the autocatalytic curve (i.e., with no maximum or minimum points and only one point of inflection) except that it is free from the two restrictive features mentioned in our first paper, that is, location of the point of inflection in the middle and symmetry of the two limbs of the curve. Asymmetrical or skew curves of this sort can only arise when the equation, $F'(x) = 0$, has no real roots. While any odd value of n may yield this form of curve the simplest equation that will do it is that in which $n = 3$, so that the equation of this curve becomes that of (vii).

Having determined that the growth within any one epoch or cycle may be approximately represented by equation (i), or more accurately by (vii), the next question is that of treating several epochs or cycles. Theoretically, some form of (v) may be found by sufficient labor in the adjustment of constants so that one equation with say 5 or 7 constants would describe a long history of growth involving several cycles. Practically, however, we have found it easier and just as satisfactory in other respects to treat each cycle by itself. Since the cycles of any case of growth are additive, we may use for any single cycle the equation

$$y = d + \frac{k}{1 + me^{ka'x}} \tag{viii}$$

or more generally

$$y = d + \frac{k}{1 + me^{a_1x + a_2x^2 + a_3x^3}} \tag{ix}$$

where in both of these forms d represents the total growth attained in all the previous cycles. The term d is therefore the lower asmptote of the cycle of growth under consideration and $d + k$ is its upper asymptote.

In treating any two adjacent cycles, it should be noted that the lower asymptote of the second cycle is frequently below the upper asymptote of the first cycle, due to the fact that the second cycle is often started before the first one has had time to reach its natural level. This for instance would be the case where a population entered upon an industrial era before the country had reached the limit of population possible under purely agricultural conditions.

The theory presented in this paper has been found to be entirely successful in fitting the population growth of many different countries, and in a subsequent publication this fact will be demonstrated with examples.

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[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, The Johns Hopkins University, No. 81.

[2] Pearl, R. and Reed, L. J., "On the Rate of Growth of the Population of the United States Since 1790 and Its Mathematical Representation." PROC. NAT. ACAD. SCI., Vol. 6, pp. 275–288, 1920.

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